MA6024 - Partial Differential Equations

Day 2/44 : Industry Problems and Million Dollar Question

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Big Three PDEs

Big Three PDEs

- Heat Equation

$$u_t = \alpha^2 u_{xx}$$

- Wave Equation

$$u_{tt} = c^2 u_{xx}$$

- Laplace Equation

$$u_{xx} + u_{yy} = 0$$

In all problems in this presentation, valid domain for time, space and function space are assumed.

Heat Equation

Heat equation has different names in different engineering and science stream.

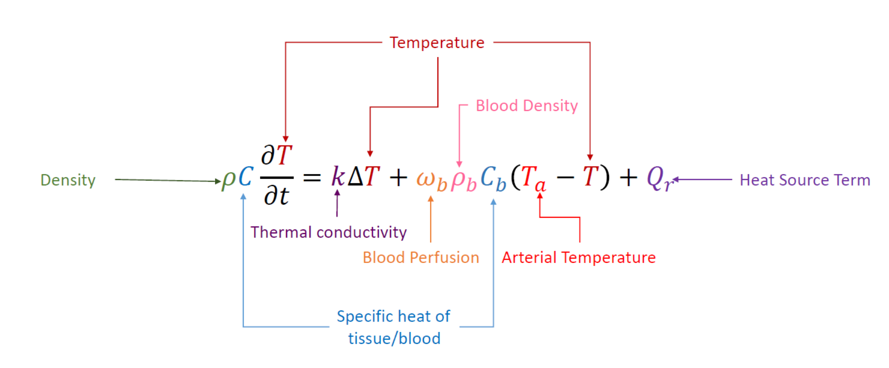
- Thermodynamics: Heat Equation
- Mechanical Engineer: Diffusion Equation
- Civil engineers (Terzaghi's theory): Consolidation equation for drilling
- Electrical Engineers: Telegraph equation
- Financial Mathematics: Black-Scholes equation

For mathematicians, it is simply a second order PDE. 😐

Heat Equation

Heat Equation with heat source, convection term are applied in the field of

- Cancer Treatment: Bioheat Equation [MWF+17]



$$\rho C \frac{\partial T}{\partial t} = k \Delta T + \omega_b \rho_b C_b (T_a - T) + Q_r$$

Heat Equation

Heat Equation is also applied in

- Medical imaging: Image smoothing [APT06]

$$\nabla^2 I = \frac{\phi''(S/\epsilon)}{\epsilon^2} \nabla S \otimes \nabla S + \frac{\phi'(S/\epsilon)}{\epsilon} \nabla^2 S \tag{1}$$

- Inverse Problem: In DCIS-Breast Cancer

$$c\frac{\partial \sigma}{\partial t} = \frac{\partial^2 \sigma}{\partial x^2} - \lambda(x)\sigma \tag{2}$$

Diffusion Equation

Atmospheric Diffusion Equation: The dispersion of pollutant concentration from multiple point sources [OP19]

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = k_x\frac{\partial^2 c}{\partial x^2} + k_y\frac{\partial^2 c}{\partial y^2} + k_z\frac{\partial^2 c}{\partial z^2} + s \qquad (3)$$

In Chemical Engineering for concentration around the point (x, y, z)

$$\frac{\partial c}{\partial t} = k_x\frac{\partial^2 c}{\partial x^2} + k_y\frac{\partial^2 c}{\partial y^2} + k_z\frac{\partial^2 c}{\partial z^2} \qquad (4)$$

Wave Equation

Lighthill's acoustic wave equation[Lig50]

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2\nabla^2\right)\rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{5}$$

(6)

$$T_{ij} = \rho u_i u_j - \tau_{ij} + [(p - p_0) - c_0^2(\rho - \rho_0)]\delta_{ij}$$

A few more ⋯

The following are a few important PDEs which arise from physical problems

- Convection or Advection or Transport

$$u_t + cu_x = 0$$

- Burgers's Equation (Dynamic Gases)

$$u_t + uu_x = 0 \quad \text{or} \quad u_t + uu_x = \mu u_{xx}, t > 0, x \in \mathbb{R}$$

- Eikonal Equation

$$u_x^2 + u_y^2 = 1$$

A few more $\cdots$

The following are a few important PDEs which arise from physical problems

- Shock Waves

$$u_x + uu_y = 0$$

- Biharmonic

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$$

- Wave equation with interaction

$$u_{tt} = u_{xx} - u^3$$

A few more ⋯

The following are a few important PDEs which arise from physical problems

- Born-Infeld

$$(1 - u_t^2)u_{xx} + 2u_x u_t u_{xt} - (1 + u_x^2)u_{tt} = 0$$

- Monge-Ampere

$$u_{xy}^2 - u_{xx}u_{yy} = f(x, y)$$

- Klein-Gordon

$$u_{tt} - c^2\nabla^2 u + \frac{m^2c^4u}{\hbar^2} = 0$$

- Hamilton-Jacobi

$$-u_t = H(q, u_q, t)$$

A few more ⋯

The following are a few important PDEs which arise from physical problems

- Schrodinger's Equation(time independent,Quantum mechanics)

$$iu_t = -\frac{\hbar}{2m}\Delta u + V(x)u(t,x) = 0, t > 0, x \in \mathbb{R}$$

- Euler-Bernoulli Beam Equation

$$u_{tt} + \alpha^4 u_{xxxx} = 0, t > 0, x \in \mathbb{R}$$

- Korteweg-de Vries Equation

$$u_t + u_{xxx} + uu_x = 0, t > 0, x \in \mathbb{R}$$

A few more ⋯

The following are a few important PDEs which arise from physical problems

- Benjamin-Bona-Mahony equation

$$u_t + u_{txx} + uu_x = 0, t > 0, x \in \mathbb{R}$$

- Vlasov-Poisson equation

$$f_t + v.\nabla_x f + E.\nabla_v f = 0, t > 0, x \in \mathbb{R}^n, v \in \mathbb{R}^n$$

$$E = \nabla_x V, \Delta V = \int_{\mathbb{R}^n} f(t,x,v)dv, V = V(x), f \geq 0$$

Maxwell's equation

$$\nabla.\mathbf{D} = \rho_v$$

$$\nabla.\mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{H}}{\partial t} + \mathbf{J}$$

Sloshing Dynamics

The following equation are used in hydrodynamic analysis of partially filled liquid tanks [HDWA19]

$$\Delta u = 0 \tag{7}$$

$$\nabla . n - \nabla \eta . \nabla u = \eta_t \tag{8}$$

$$t + g\eta + \frac{1}{2}(\nabla u)^2 = 0 \tag{9}$$

(10)

Shallow Water Equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\Big((H+h)u\Big) + \frac{\partial}{\partial y}\Big((H+h)v\Big) = 0, \tag{11}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} - bu + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{12}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} - bv + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{13}$$

Reynolds Transport

Reynolds Transport Theorem

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v.\nabla f$$

Euler equation

$$\frac{D\rho}{Dt} = 0$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} + \mathbf{g}$$

$$\nabla . \mathbf{u} = 0$$

Navier-Stokes

Continuity Equation (Conservation of Mass)

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0 \tag{14}$$

Momentum Equation (Conservation of Momentum)

$$\frac{\partial}{\partial t}(\rho\,\mathbf{u}) + \nabla \cdot (\rho\,\mathbf{u} \otimes \mathbf{u}) = -\nabla \bar{p} + \mu\,\nabla^2 \mathbf{u} + \tfrac{1}{3}\mu\,\nabla(\nabla \cdot \mathbf{u}) + \rho \mathbf{g} \tag{15}$$

Energy Equation (Conservation of Energy)

$$\frac{\partial}{\partial t}(\rho\,\mathbf{e}) + \nabla.[\rho \mathbf{u}\mathbf{e} + \dot{q}_s - \tau.\mathbf{u}] = p\nabla.\mathbf{u} + \mathbf{f_b}.\mathbf{u} + \dot{q}_u \tag{16}$$

Weather Prediction

Momentum: $$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{V}.\nabla \mathbf{V} + 2 \times \mathbf{V} + \frac{1}{\rho}\nabla p = \mathbf{F} + \mathbf{g} \quad (17)$$

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0 \quad (18)$$

Water substance: $$\frac{dq}{gt} = S \quad (19)$$

Hydrostatic: $$\frac{\partial p}{\partial z} + g\rho = 0 \quad (20)$$

Drones/Helicopter Rotor

$$(\Omega.\nabla)\mathbf{v} = \frac{D\Omega}{Dt} \quad (21)$$

$$\frac{\partial \Omega}{\partial t} + \mathbf{v}.\nabla = (\Omega.\nabla)\mathbf{v} \quad (22)$$

$$\Omega = \nabla \times \mathbf{v} \quad (23)$$

Minimal Surface

$$(1+|\nabla u|^2)\Delta u - \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

Goals

When an industrial problem or real-world or physical problem is posted to you, you need to model them mathematically. In case, your mathematical model falls under the scope of partial differential equations. Then one of the fundamental question is "Which PDEs should I choose?". No general answer!

- Discussion of some important physical system (dated back in 18th and 19th Century)
- Existence and Uniqueness

Goals

Let us try to answer the following questions in this lecture:

- Does the PDE have any solutions?
- What are all necessary conditions to solve a PDE?
- Are the solutions unique for a given data?
- What are the basic qualitative properties of the solution?
- What is the nature of singularities, if any?
- How does a small perturbation in the input data affect the solution?
- What types of quantitative estimates can be obtained for a given solution?
- How can we define norm of a solution and find their error estimates with respect to norm?

Clay Math Problem: Million Dollar Question

Universal Solution??

- It is not necessary that all PDEs have solution. For example, $u_x^2 + 1 = 0$ has no solution.
- We can't expect a general existence theorem for a general PDE
- Remember: We do not expect solution for all PDES as we do in linear algebra or matrix in school days.
- Recall: All polynomials or system of linear equations or implicit functions do not have solutions always.
- From Linear Algebra/Matrix Theory: Under certain conditions, linear system of equations have solution(s). We characterize them depending on the rank of the matrix and the corresponding augmented matrix
- How about non-linear equations?

Universal Solution??

In ODE course:

- Peano's existence theorem
- Picard's existence and uniqueness theorem
- These theorems address existence of solutions of IVP for first order ODE
- Extends this to study any ODE in normal form

In PDE:

- Can you expect a similar one for PDE? Not possible
- Different types of problems have different conditions
- Let us see one of the million dollar question

Clay Mathematics

- Clay Mathematics founded by T.Clay and his wife in 1998 stated 7 unsolved problems in 2000 called Millennium (Prize) Problems
- Correct solutions discoverer of any of these 7 problems will get 1 Million US dollar
- Poincare Conjecture is the only problem solved so far (by Perelman, 2003, but he declined the money)
- Navier-Stokes existence and smoothness is one of the millennium problem in the remaining 6 unsolved problems.

Source: Rest of the contents for this presentation is taken from the following link: http://claymath.org/sites/default/files/navierstokes.pdf

Navier-Stokes

Consider the following three-dimensional Navier-Stokes equation

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= -p_x + \nu(u_{xx} + u_{yy} + u_{zz}) + f_1(x, y, z, t) \\ v_t + uv_x + vv_y + wv_z &= -p_y + \nu(v_{xx} + v_{yy} + v_{zz}) + f_2(x, y, z, t) \\ w_t + uw_x + vw_y + ww_z &= -p_z + \nu(w_{xx} + w_{yy} + w_{zz}) + f_3(x, y, z, t) \\ u_x + v_y + w_z &= 0 \end{aligned} \tag{24}$$

It is a second-order PDE with four variables $u(x, y, t, z)$, $v(x, y, z, t)$,$w(x, y, z, t)$, $p(x, y, z, t)$, ν kinematic viscosity of the fluid. f_is are external force. We restrict to incompressible (ρ is constant) fluids which is represented by the last equation. $\nu = 0$ implies Euler equations.

Navier-Stokes

- The Navier-Stokes equations are fundamental equation in fluid mechanics.
- Difficult to solve either analytically or numerically
- When the initial condition $u(x, y, z, 0) = u_0(x, y, z)$ is given finding the existence or nonexistence of solutions for all future times is a major unresolved problem in mathematics

Navier-Stokes

Let $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$,
$\mathbf{u}(\mathbf{x}, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)) \in \mathbb{R}^3$, $\mathbf{p}(\mathbf{x}, t) \in \mathbb{R}$
and $\mathbf{f}(\mathbf{x}, t) = (f_1(x, y, z, t), f_2(x, y, z, t), f_3(x, y, z, t)) \in \mathbb{R}^3$.

Navier-Stokes

For physically reasonable, we would like to make sure $\mathbf{u}(\mathbf{x}, t)$ does not grow large as $|\mathbf{x}| \to \infty$. Let us keep some restrictions on $\mathbf{f}$ and $\mathbf{u_0}$.

$$|\partial_{\mathbf{x}}^{\alpha} \mathbf{u_0}(\mathbf{x})| \leq C_{\alpha K}(1+|x|)^{-K}, \quad \text{on} \quad \mathbb{R}^3, \quad \text{for any} \quad \alpha, K \tag{25}$$

$$|\partial_{\mathbf{x}}^{\alpha} \partial_t \mathbf{f}(\mathbf{x}, t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}, \quad \text{on} \quad \mathbb{R}^3, t \geq 0 \quad \text{for any} \quad \alpha, m, K \tag{26}$$

Solution of (22) is accepted as physically reasonable only if it satisfies

$$\mathbf{p}, \mathbf{u} \in C^{\infty}(\mathbb{R}^3, [0, \infty)) \tag{27}$$

$$\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x} < C, \quad \text{for all} \quad t \geq 0 \tag{28}$$

Navier-Stokes

If problems at infinity are ruled out, we can look for spatially periodic solutions. Therefore, restriction on **f** and $\mathbf{u_0}$ becomes

$$\mathbf{u_0}(\mathbf{x}+\mathbf{e_j}) = \mathbf{u_0}(x), \ \ \mathbf{f}(\mathbf{x}+\mathbf{e_j}, t) = \mathbf{f}(x,t), 1 \le j \le 3 \tag{29}$$

$$|\partial_{\mathbf{x}}^{\alpha}\partial_t \mathbf{f}(\mathbf{x},t)| \le C_{\alpha m K}(1+t)^{-K}, \ \text{ on } \ \mathbb{R}^3, t \ge 0 \ \text{ for any } \ \alpha, m, K \tag{30}$$

Solution of (22) is accepted as physically reasonable only if it satisfies

$$\mathbf{p}, \mathbf{u} \in C^{\infty}(\mathbb{R}^3, [0,\infty)) \tag{31}$$

$$\mathbf{u}(\mathbf{x}+\mathbf{e_j}, t) = \mathbf{u}(x,t) \ \text{ on } \ \mathbb{R}^3 \times [0,\infty) \ \text{ for } \ 1 \le j \le 3 \tag{32}$$

Millennium Problem

Unsolved Problems (Existence and Uniqueness of NS Solution-A)

Let $\nu > 0$. Let $\mathbf{u_0}(\mathbf{x})$ be any smooth, that is, $u_0 \in C^\infty(\mathbb{R}^3)$, divergence free vector field which satisfies the condition (23) Let $\mathbf{f} = 0$. Then there exist smooth functions $\mathbf{p}(\mathbf{x}, t), u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy the above (22), (25) and (26).

Millennium Problem

Unsolved Problems (Existence and Uniqueness of NS Solution-B)

Let $\nu > 0$. Let $\mathbf{u_0}(\mathbf{x})$ be any smooth, that is, $u_0 \in C^\infty(\mathbb{R}^3)$, divergence free vector field which satisfies the condition (27) Let $\mathbf{f} = 0$. Then there exist smooth functions $\mathbf{p}(\mathbf{x}, t), u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy the above (22), (29) and (30).

Millennium Problem

Unsolved Problems (Breakdown of NS Solution-C)

Let $\nu > 0$*. Then there exist a smooth, divergence free vector field* $\mathbf{u_0}(\mathbf{x})$*, that is* $u_0 \in C^\infty(\mathbb{R}^3)$*, and a smooth* $\mathbf{f}(\mathbf{x}, \mathbf{t})$ *on* $\mathbb{R}^3 \times [0, \infty)$ *satisfying (23) and (24) for which there exist no solutions* $(\mathbf{p}, \mathbf{u})$ *of (22), (25) and (26) on* $\mathbb{R}^3 \times [0, \infty)$

Millennium Problem

Unsolved Problems (Breakdown of NS Solution-D)

Let $\nu > 0$*. Then there exist a smooth, divergence free vector field* $\mathbf{u_0}(\mathbf{x})$*, that is* $u_0 \in C^\infty(\mathbb{R}^3)$*, and a smooth* $\mathbf{f}(\mathbf{x}, \mathbf{t})$ *on* $\mathbb{R}^3 \times [0, \infty)$ *satisfying (27) and (28), for which there exist no solutions* $(\mathbf{p}, \mathbf{u})$ *of (22), (29) and (30) on* $\mathbb{R}^3 \times [0, \infty)$

References I

- S. Angenent, E. Pichon, and A. Tannenbaum, *Mathematical methods in medical image processing*, 2006, pp. 365–396.
- M. Han, J. Dai, C.M. Wang, and K. K. Ang, *Hydrodynamic analysis of partially filled liquid tanks subject to 3d vehicular manoeuvring*, 2019, pp. 1–14.
- M. J. Lighthill, *On sound generated aerodynamically. i. general theory*, 1950, pp. 564–587.
- Panchatcharam M, Phil W, Ronan F, Philip V, TuomasA, Mika P, Michael M, Harald B, Rupert H P, Roberto B S, and Marina K, *Gpu-based rfa simulation for minimally invasive cancer treatment of liver tumours*, 2017, pp. 59–68.
- P. Oyjinda and N. Pochai, *Numerical simulation of an air pollution model on industrial areas by considering the influence of multiple point sources*, 2019, p. 10.