MA6024-Partial Differential Equations

Day 4/44: Simple Problems, Boundary Conditions, and Method of Characteristics

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Classification of PDEs (Recap)
Definition 1 (PDE-Formal Definition)
Let $\Omega \subset \mathbb{R}^n$, $m \in \mathbb{N}$ and

$$F : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \mathbb{R}^{n^2p} \times \cdots \times \mathbb{R}^{n^mp} \rightarrow \mathbb{R}^q$$

A system of partial differential equations of order $m$ is defined by the equation

$$F(x, u, Du, D^2u, \cdots, D^m u) = 0$$  \hspace{1cm} (1)

Here some $m^{th}$ order derivative of the function $u$ appears in the system of equations.
Definition 1 (PDE-Formal Definition)
Let \( \Omega \subset \mathbb{R}^n, m \in \mathbb{N} \) and

\[
F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^m} \to \mathbb{R}
\]

The partial differential equation of order \( m \) is defined by the equation

\[
F(x, u, Du, D^2u, \cdots, D^m u) = 0
\]

(2)

Here some \( m^{th} \) order derivative of the function \( u \) appears in the equation and \( u : \Omega \to \mathbb{R} \)
Definition 2
If a PDE (1) consists of more than one equation, it is called a **system of PDEs**. Otherwise, it is called a single PDE or a scalar PDE or simply PDE.

Definition 3
If the highest order derivative appearing in the PDE is $m$, then such PDEs are classified as $m^{th}$ order PDEs.
Rewrite the equation (2) as

$$\mathcal{L}u = f$$  \hspace{1cm} (3)

where $\mathcal{L}$ is an operator. The operator $\mathcal{L}$ is called linear if

$$\mathcal{L}(\alpha u + \beta v) = \alpha \mathcal{L}u + \beta \mathcal{L}v$$

for any function $u$ and $v$ and any constants $\alpha$ and $\beta$.

**Definition 4**

If the operator $\mathcal{L}$ in (3) is linear, then the PDE is called **linear PDE**. Equivalently, if is of the form

$$\sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u = f(x)$$  \hspace{1cm} (4)

If the operator $\mathcal{L}$ in (3) is not linear (or equivalently it can’t be written in the form of (4)), then the PDE is called **nonlinear** PDE.
**Definition 5**

The equation (2) of order $m$ is called **quasilinear** if it is linear in the derivatives of order $m$ with coefficients that depend on the independent variables and derivatives of the unknown function of order strictly less than $m$. Equivalently,

$$
\sum_{|\alpha|=m} a_{\alpha}(x, u, Du, \cdots D^{m-1}u)D^\alpha u + a_0(x, u, Du, \cdots D^{m-1}u) = 0
$$

(5)

An $m^{th}$ order PDE is called **fully nonlinear** if it is not linear in the derivatives of order $m$. Equivalently, a PDE which is not quasilinear is called fully nonlinear PDE.
Definition 6
A quasilinear PDE of order $m$ is called a **semilinear PDE** if the coefficients of derivatives of order $m$ are functions of the independent variables alone. Equivalently

$$
\sum_{|\alpha|=m} a_\alpha(x) D^\alpha u + a_0(x, u, Du, \cdots D^{m-1}u) = 0
$$

Here $a_\alpha$’s are function of $x$ alone.
Definition 7
An $m^{th}$ order semilinear PDE is called almost linear if it can be written in the form

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u + f(x, u) = 0$$  \hspace{1cm} (7)

Here $a_\alpha$’s are function of $x$ alone or if it is of the form

$$\mathcal{L}u = f(x, u)$$  \hspace{1cm} (8)

where $f(x, u)$ is a nonlinear function with respect to $u$ and $\mathcal{L}$ is a linear operator.

Example 2
1. $u_t + u_x + u^2 = 0$ is almost linear
2. $xu_x + yu_y = u$ is almost linear
Classification-VII - In/Homogeneous

Suppose (1) can be written in the following form

\[ D(u) = f(x) \]  \hspace{1cm} (9)

**Definition 8**
If \( f \equiv 0 \) in (7), then the PDE is called homogeneous PDE. If \( f \neq 0 \), then the PDE is inhomogeneous PDE\(^1\).

**Example 3**
1. \( u_t + uu_x = 0 \) is homogeneous
2. \( 2u_y - 5u^3 = x \) is inhomogeneous
3. \( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = f(r, \theta) \) is inhomogeneous if \( f \neq 0 \)

\(^1\)In some text books it is also called nonhomogeneous PDE. Also, many text books usually classify only linear PDE as homogeneous and nonhomogeneous.
Examples

Example 4

<table>
<thead>
<tr>
<th>PDE</th>
<th>O</th>
<th>Lin</th>
<th>AL</th>
<th>Sem</th>
<th>Qua</th>
<th>HG</th>
<th>FNL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_t + u_x + u^2 = 0)</td>
<td>1</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>(u_{xx} + u_{yy} = 0)</td>
<td>2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
</tr>
<tr>
<td>(u_x^2 + u_y^2 = x^2 + y^2)</td>
<td>1</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>(u_x + 5u = x^2y)</td>
<td>1</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>×</td>
<td>×</td>
</tr>
</tbody>
</table>

O - Order, Lin - Linear, AL - Almost linear, Sem - Semilinear, Qua - Quasilinear, HG - Homogeneous, FNL - Fully nonlinear.
First-order PDEs in Two Variables

The general first order PDEs in two variables can be written in the form

\[ F(x, y, u, u_x, u_y) = 0 \]  \hspace{2cm} (10)

The first-order **linear** PDE is of the form

\[ a(x, y)u_x + b(x, y)u_y = c(x, y)u + f(x, y) \] \hspace{2cm} (11)

The first-order **semilinear** PDE is of the form

\[ a(x, y)u_x + b(x, y)u_y = c(x, y, u) \] \hspace{2cm} (12)

The first-order **quasilinear** PDE is of the form

\[ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \] \hspace{2cm} (13)
Second-order PDEs in Two Variables

The general second-order PDEs in two variables can be written in the form

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \]

The second-order \textbf{linear} PDE is of the form

\[ a_1(x, y)u_{xx} + a_2(x, y)u_{xy} + a_3(x, y)u_{yy} + a_4(x, y)u_x + a_5u_y + a_6(x, y)u = f(x, y) \]

The second-order \textbf{semilinear} PDE is of the form

\[ a_1(x, y)u_{xx} + a_2(x, y)u_{xy} + a_3(x, y)u_{yy} = f(x, y, u, u_x, u_y) \]

The second-order \textbf{quasilinear} PDE is of the form

\[ a_1(x, y, u, u_x, u_y)u_{xx} + a_2(x, y, u, u_x, u_y)u_{xy} + a_3(x, y, u, u_x, u_y)u_{yy} = f(x, y, u, u_x, u_y) \]
Classification

Remarks 1
A few authors classify

- only linear PDE as homogeneous and nonhomogeneous
- nonlinear PDE as semilinear and non-semilinear
- non-semilinear PDE as quasilinear and non-quasilinear/fully nonlinear

Later, we will see some more classifications like parabolic, elliptic and hyperbolic when we discuss second-order PDE which can be extended further for higher-order PDEs.

Remarks 2
One can prove that

Linear PDE ⊂ Semilinear PDE ⊂ Quasilinear PDE ⊂ PDE

(Prove that the inclusion is strict!)
Exercise

Exercise 1: Hard

Create a table like Example 7 and fill the tick marks accordingly.

1. $u_{xxx} - 4u_{xxyy} + u_{yyzz} = 0$
2. $u_x^2u_{tt} - 0.5u = 1 - u^2$
3. $u_{tt}u_{xxx} - u_xu_{ttt} = x^2 + t^2$
4. $e^{u_{xxx}} - u_xtu_{xxx} + u^2 = 0$
5. $2\cos(xt)u_t - xe^t u_x - 9u = e^t \sin x$
6. $uu_t + u^2u_x + u = e^x$
7. $\sqrt{1 + x^2y^2}u_{xyy} - \cos(xy^3)u_{xxy} + e^{-y^3}u_x - (5x^2 - 2xy + 3y^2)u = 0$
Preliminaries
for Boundary Conditions and Method of Characteristics

Vector Differential Operators
Directional Derivative

Recall the directional derivative definition

**Definition 9 (Directional Derivative)**

Let \( f(x, y) \) be a function defined in a domain \( \Omega \subset \mathbb{R}^2 \). Let \((x_0, y_0) \in \Omega\). The **directional derivative** of \( f(x, y) \) in the direction of a unit vector \( \mathbf{v} = ai + bj \) at \((x_0, y_0)\) is given by

\[
(D_\mathbf{v} f)(x_0, y_0) = \left(\frac{df}{ds}\right)_{\mathbf{v}} \bigg|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}
\]

Here \( D_\mathbf{v} \) denotes the directional derivative in the direction of \( \mathbf{v} \)

From Rudin: If \( \mathbf{v} = \sum v_i \mathbf{e}_i \), then

\[
D_\mathbf{v} f(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x)v_i
\]
Gradient

**Definition 10 (Gradient)**

The vector operator

\[
\nabla \equiv \sum_{i=1}^{n} \frac{\partial}{\partial x_i} e_i
\]

is called the gradient. The gradient of a function \( f(x_1, x_2, \cdots, x_n) \) is

\[
\nabla f \equiv \text{grad} f \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \)

In 2D case,

\[
\nabla f \equiv \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}
\]
Gradient

Remark

\[ D_v f \big|_{(x_0, y_0)} = \text{grad} f \big|_{(x_0, y_0)} \cdot v \]
\[ D_v f (x) = \nabla f \cdot v \]

If \( v = a \mathbf{i} + b \mathbf{j} \),

\[ D_v f = f_x a + f_y b = (f_x \mathbf{i} + f_y \mathbf{j}) \cdot (a \mathbf{i} + b \mathbf{j}) \]

If \( D_v f = 0 \), then \( f \) is constant in the direction of the vector \( v \). The gradient of a function at a point is a vector that points in the direction in which the function increase most rapidly.
Divergence

**Definition 11 (Divergence)**

The divergence of a vector field is the flux per unit time. It is defined as an inner product between the gradient operator and the vector field

\[
\nabla \cdot \mathbf{v} \equiv \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}
\]

where \( \mathbf{v} : \mathbb{R}^n \rightarrow \mathbb{R}^n \)

In 3D, if \( \mathbf{v} = (v_1, v_2, v_3) \), then

\[
\nabla \cdot \mathbf{v} \equiv \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}
\]
**Laplacian**

**Definition 12 (Laplacian)**
The Laplacian of a scalar valued function is defined as

\[ \Delta f \equiv \nabla^2 f = \nabla \cdot (\nabla f) = \sum_{i=1}^{n} \frac{\partial^2 v_i}{\partial x_i^2} \]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R} \)

**Definition 13 (Laplacian of a vector)**
The Laplacian of a vector valued function is defined as

\[ \Delta \mathbf{v} \equiv \nabla^2 \mathbf{v} = \nabla \cdot (\nabla \mathbf{v}) = \sum_{i=1}^{n} \frac{\partial^2 v_i}{\partial x_i^2} \mathbf{e}_i \]

where \( \mathbf{v} : \mathbb{R}^n \rightarrow \mathbb{R}^n \)
Boundary Conditions
Definition 14 (Initial Value Problem)
A partial differential equation subject to certain conditions in the form of initial conditions is known as **initial value problem** or in short IVP. Usually the initial conditions are given as $u(x, t_0) = f(x)$.

Example 5

$$
\begin{cases}
    u_t - u_x = 0, & -\infty < x < \infty, t > 0 \\
    u(x, 0) = \phi(x), & -\infty < x < \infty, t = 0
\end{cases}
$$
Definition 15 (Boundary Value Problem)
A partial differential equation subject to certain conditions in the form of boundary conditions is known as **boundary value problem** or in short BVP. Usually the boundary conditions are given as the values on the boundary $\partial \Omega$

Example 6

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0, \quad (x, y) \in \Omega \\
    u(x, y) &= \phi(x, y), \quad (x, y) \in \partial \Omega
\end{align*}
\]
Dirichlet Boundary Condition

There are three types of boundary conditions usually prescribed (although others conditions, like periodic, inlet, outlet are also available) **Dirichlet Boundary Condition**: The solution known at the boundary of the domain or the values of $u$ are prescribed at each point of the boundary $\partial \Omega$

$$u(x, t) = f(x), x \in \partial \Omega, t > 0$$

**Example 7**

\[
\begin{align*}
\begin{cases}
uxx + uyy &= 0, \quad (x, y) \in \Omega \\
u(x, y) &= \phi(x, y), \quad (x, y) \in \partial \Omega
\end{cases}
\end{align*}
\]

This is also called as **Fixed or Essential Boundary Condition** or boundary conditions of the first kind.
**Neumann Boundary Condition**

**Neumann Boundary Condition**: The derivative of the solution is given in a direction at the boundary of the domain or the values of the normal derivative of $u$ are prescribed at each point of the boundary $\partial \Omega$

$$\frac{\partial u}{\partial n} = n \cdot \nabla u = f(x), \quad x \in \partial \Omega$$

Here $n = n(x)$ is the outward unit normal to $\partial \Omega$ at $x \in \partial \Omega$

**Example 8**

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0, \quad (x, y) \in \Omega \\
    \frac{\partial u}{\partial n} &= \psi(x, y), \quad (x, y) \in \partial \Omega
\end{align*}
\]

This is also called as **Natural Boundary Condition** or boundary conditions of the second kind.
Robin Boundary Condition

Robin Boundary Condition: It is a linear combination of Dirichlet and Neumann boundary conditions or when the values of a linear combination of $u$ and its normal derivative are prescribed at each point of the boundary $\partial \Omega$

$$\alpha \frac{\partial u}{\partial n} + \beta u(x) = f(x), \ x \in \partial \Omega$$

Example 9

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in \Omega \\ \alpha \frac{\partial u}{\partial n} + \beta u = g(x, y), & (x, y) \in \partial \Omega \end{cases}$$

This is also called as **impedance or convective boundary condition** or boundary conditions of the third kind.
Well-Posedness and Solution Methods
The study of PDEs are categorized into Theory and Applications. In Theory, we study the following:

1. Existence of Solutions: To check whether at least one solution exists for a given PDE
2. Uniqueness: To study whether at most one solution by imposing conditions
3. Stability: Continuous dependence of data

In applications, we study the following:

1. Solution construction methods
2. Smoothness of the solution or Regularity
3. Approximate solution whenever the exact solution construction is not possible
Well-Posed Problem in the sense of Hadamard

A problem (PDE with Initial and Boundary Conditions) is said to be **well-posed** if it satisfies the following criteria

1. **Existence**: The solution must exist
2. **Uniqueness**: The solution should be unique
3. **Stability**: The solution should depend continuously on the initial and/or boundary data

If one or more of the conditions above does not hold, we say that the problem is **ill-posed**.
Solution of PDE

We desired to solve PDE such that the criteria of well-posedness satisfies.

**What is a solution?**

- A function $u(x)$ is a solution to (??) if $u$ and its partial derivatives appearing in (??) satisfies (??) identically for $x \in \Omega$.
- Should it be real analytic? Should $u \in C^\infty(\Omega)$?
- We wish! But we are expecting too much.
- It is better to require a solution $u$ of an $m$th order PDE such that $u \in C^m(\Omega)$. (Why?)
- We call a solution with this much smoothness a classical solution of the PDE.
Question: Can we always obtain a classical solution.

Answer: Not always, for certain specific PDEs we can obtain. For example, consider the Burger’s equation

$$u_t + uu_x = 0$$

This models the formation and propagation of shock waves. A **Shock wave** is a curve of discontinuity of a solution $u$. To study conservation laws and underlying physics, you must allow $u$ as not continuously differentiable, some times not continuous also. Hence, conservation laws in general do not possess classical solution. However, if we allow weak solutions or generalized solutions, it is well-posed.
Methods to solve PDEs

In this course, we will see mostly the following approaches to solve PDEs

1. **Fourier Method**: We apply the **Separation of variable** methods to reduce the PDEs to simpler eigenvalue problems in ODEs. This method is called as Fourier method or Separation of Variables method or solution by eigenfunction expansion method.

2. **Green’s Function Method**: We apply the fundamental solution to search for radial solutions. Also known as Green’s function method.

3. **Energy Method**: Using calculus of variation method or Variational formulation method

4. **Method of Characteristics

5. **d’Alembert’s Method**
Thanks

Doubts and Suggestions

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