भारतीय प्रौद्योगिकी संस्थान तिरुपति

TIRUPATI

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DEPARTMENT OF MATHEMATICS AND STATISTICS

MA6024-PARTIAL DIFFERENTIAL EQUATIONS

Assignment-2

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1. Consider the following wave equation

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = 0 \ \text{ in } \ \mathbb{R}^2 \times (0, \infty) \\ u(x, y, 0) = f(x, y) \ \text{ on } \ \mathbb{R}^2 \times \{t = 0\} \\ u_t(x, y, 0) = g(x, y) \ \text{ on } \ \mathbb{R}^2 \times \{t = 0\} \end{cases} \tag{1}$$

Let $f \in C^3(\mathbf{R}^2)$ and $g \in C^2(\mathbb{R}^2)$. Prove that the solution of the above wave equation is given by

$$u(x, y, t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int_{B(x,y,ct)} \frac{f(x_1, y_1)}{\sqrt{c^2t^2 - (x - x_1)^2 - (y - y_1)^2}} dx_1 dy_1\right) + \frac{1}{2\pi c}\int_{B(x,y,ct)} \frac{g(x_1, y_1)}{\sqrt{c^2t^2 - (x - x_1)^2 - (y - y_1)^2}} dx_1 dy_1$$

[30]

2. Consider the following wave equation

$$\begin{cases} u_{tt} - c^2(u_{xx}) = h(x, t) \ \text{ in } \ \mathbb{R} \times (0, T) \\ u(x, 0) = f(x) \ \text{ on } \ \mathbb{R} \times \{t = 0\} \\ u_t(x, 0) = g(x) \ \text{ on } \ \mathbb{R} \times \{t = 0\} \end{cases} \tag{2}$$

Let $T > 0, h \in C(\mathbb{R} \times [0, T])$ and $\frac{\partial h}{\partial x} \in C(\mathbb{R} \times [0, T])$. Also, $f, g \in C^2(\mathbb{R})$.

(a) Prove that there exists $u \in C^2(\mathbb{R} \times (0, T)) \cap C^1(\mathbb{R} \times [0, T])$ such that u satisfies equation (2). [15]

(b) There exists a unique solution for (2) [5]

(c) Given $\epsilon > 0$, there exists $\delta > 0$ such that for every pair of data triples (h_1, f_1, g_1) and (h_2, f_2, g_2) having the smoothness and satisfies the following condition

$$|f_1(x) - f_2(x)| < \delta \quad \text{for all} \quad x \in \mathbb{R}$$

$$|g_1(x) - g_2(x)| < \delta \quad \text{for all} \quad x \in \mathbb{R}$$

and

$$|h_1(x,t) - h_2(x,t)| < \delta \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0,T]$$

then prove that the corresponding solutions u_1 and u_2 of (2) satisfies the following condition

$$|u_1(x,t) - u_2(x,t)| < \epsilon \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0,T]$$

[10]

3. Consider the following wave equation

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = h(x,y,t) \text{ in } \mathbb{R}^2 \times (0,T) \\ u(x,y,0) = f(x,y) \text{ on } \mathbb{R}^2 \times \{t = 0\} \\ u_t(x,y,0) = g(x,y) \text{ on } \mathbb{R}^2 \times \{t = 0\} \end{cases} \tag{3}$$

Let $T > 0, h \in C(\mathbb{R}^2 \times [0,T])$ and $\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) \in C(\mathbb{R}^2 \times [0,T])$. Also, $f, g \in C^3(\mathbb{R}^2)$.

(a) Prove that there exists $u \in C^2(\mathbb{R}^2 \times (0,T)) \cap C^1(\mathbb{R}^2 \times [0,T])$ such that u satisfies equation (3). [15]

(b) There exists a unique solution for (3) [5]

(c) Given $\epsilon > 0$, there exists $\delta > 0$ such that for every pair of data triples (h_1, f_1, g_1) and (h_2, f_2, g_2) having the smoothness and satisfies the following condition

$$|f_1(x,y) - f_2(x,y)| < \delta \quad \text{for all} \quad x, y \in \mathbb{R}^2$$

$$|g_1(x,y) - g_2(x,y)| < \delta \quad \text{for all} \quad x, y \in \mathbb{R}^2$$

and

$$|h_1(x,y,t) - h_2(x,y,t)| < \delta \quad \text{for all} \quad (x,y,t) \in \mathbb{R}^2 \times [0,T]$$

then prove that the corresponding solutions u_1 and u_2 of (3) satisfies the following condition

$$|u_1(x,y,t) - u_2(x,y,t)| < \epsilon \quad \text{for all} \quad (x,y,t) \in \mathbb{R}^2 \times [0,T]$$

[10]

4. Consider the following wave equation

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy} + +u_{zz}) = h(x,y,z,t) \text{ in } \mathbb{R}^3 \times (0,T) \\ u(x,y,z,0) = f(x,y,z) \text{ on } \mathbb{R}^3 \times \{t = 0\} \\ u_t(x,y,z,0) = g(x,y,z) \text{ on } \mathbb{R}^3 \times \{t = 0\} \end{cases} \tag{4}$$

Let $T > 0, h \in C(\mathbb{R}^3 \times [0,T])$ and $\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right) \in C(\mathbb{R}^3 \times [0,T])$. Also, $f, g \in C^3(\mathbb{R}^3)$.

(a) Prove that there exists $u \in C^2(\mathbb{R}^3 \times (0,T)) \cap C^1(\mathbb{R}^3 \times [0,T])$ such that u satisfies equation (4). [15]

(b) There exists a unique solution for (4) [5]

(c) Given $\epsilon > 0$, there exists $\delta > 0$ such that for every pair of data triples (h_1, f_1, g_1) and (h_2, f_2, g_2) having the smoothness and satisfies the following condition

$$|f_1(x,y,z) - f_2(x,y,z)| < \delta \quad \text{for all} \quad x,y,z \in \mathbb{R}^3$$

$$|g_1(x,y,z) - g_2(x,y,z)| < \delta \quad \text{for all} \quad x,y,z \in \mathbb{R}^3$$

and

$$|h_1(x,y,z,t) - h_2(x,y,z,t)| < \delta \quad \text{for all} \quad (x,y,z,t) \in \mathbb{R}^3 \times [0,T]$$

then prove that the corresponding solutions u_1 and u_2 of (4) satisfies the following condition

$$|u_1(x,y,z,t) - u_2(x,y,z,t)| < \epsilon \quad \text{for all} \quad (x,y,z,t) \in \mathbb{R}^3 \times [0,T]$$

[10]

5. Consider the following wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } (0,1) \times (0,\infty) \\ u(x,0) = 0 \text{ on } [0,1] \times \{t=0\} \\ u_t(x,0) = x(1-x) \text{ on } (0,1) \times \{t=0\} \\ u(0,t) = 0, \{x=0\} \times [0,\infty) \\ u(1,t) = 0, \{x=1\} \times [0,\infty) \end{cases} \tag{5}$$

Find $u\left(\frac{1}{2}, \frac{3}{2}\right), u\left(\frac{1}{4}, \frac{3}{4}\right), u\left(\frac{x}{,}\frac{3}{2}\right)$ [3× 10=30]

6. Consider the following wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R} \times (0,\infty) \\ u(x,0) = \begin{cases} 0 & -\infty < x < -1 \\ x+1 & -1 \le x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & 1 < x \le \infty \end{cases} \\ u_t(x,0) = \begin{cases} 0 & -\infty < x < -1 \\ 1 & -1 \le x \le 1 \\ 0 & 1 < x \le \infty \end{cases} \end{cases} \tag{6}$$

Find $u\left(1, \frac{1}{2}\right)$,$u\left(5, \frac{5}{2}\right)$ [2×15 = 30]

7. Consider the following wave equation

$$\begin{cases} u_{tt} - 9u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u(x,0) = \begin{cases} 1 & -2 \le x \le 2 \\ 0 & |x| \ge 2 \end{cases} \\ u_t(x,0) = \begin{cases} 1 & -2 \le x \le 2 \\ 0 & |x| \ge 2 \end{cases} \end{cases} \tag{7}$$

(a) Find $u\left(0, \frac{1}{6}\right)$ [15]

(b) Find the largest subdomain of $\mathbb{R} \times (0, \infty)$ on which u is a classical solution of (3) [15]

8. Let u be a classical solution to the Wave equation $u_{tt} - u_{xx} = 0$ for $(x,t) \in \mathbb{R} \times (0, \infty)$. Prove the following

(a) $u(x - y, t)$ is also a solution for each $y \in \mathbb{R}$. [10]

(b) $\frac{\partial^k u}{\partial x^k} u(x,t)$ is also a solution for each $k \in \mathbb{N}$ [10]

(c) $u(ax, at)$ is also a solution for each $a > 0$. [10]

9. Solve the following inhomogeneous wave equation. [30]

$$\begin{cases} u_{tt} - u_{xx} = t^7 \text{ in } \mathbb{R} \times (0, \infty) \\ u(x,0) = 2x + \sin x \text{ on } \mathbb{R} \times \{t = 0\} \\ u_t(x,0) = 0 \text{ on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{8}$$

10. Solve the following wave equation. [30]

$$\begin{cases} u_{tt} - 4u_{xx} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ u(x,0) = x^2 \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ u_t(x,0) = 6x \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ u(0,t) = t^2, \quad \{x = 0\} \times (0, \infty) \end{cases} \tag{9}$$

Find $u(3, 2)$ and $u(2, 3)$

11. Solve the following wave equation. [30]

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \left(0, \frac{\pi}{2}\right) \times (0, \infty) \\ u(x,0) = \sin x \text{ on } \left[0, \frac{\pi}{2}\right] + \times \{t = 0\} \\ u_t(x,0) = 0 \text{ on } \left[0, \frac{\pi}{2}\right] \times \{t = 0\} \\ u(0,t) = t^2, \quad \{x = 0\} \times (0, \infty) \\ u_x\left(\frac{\pi}{2}, t\right) = 0 \quad \left\{x = \frac{\pi}{2}\right\} \times (0, \infty) \end{cases} \tag{10}$$

12. Solve the following damped wave equation using separation of variables [30]

$$\begin{cases} u_{tt} + 2u_t - u_{xx} = 0 \text{ in } (0,\pi) \times (0,\infty) \\ u(x,0) = f(x) \text{ on } [0,\pi] + \times\{t=0\} \\ u_t(x,0) = 0 \text{ on } [0,\pi] \times \{t=0\} \\ u(0,t) = 0, \quad \{x=0\} \times (0,\infty) \\ u(\pi,t) = 0 \quad \{x=\pi\} \times (0,\infty) \end{cases} \tag{11}$$

13. Solve the following damped wave equation using separation of variables. [30]

$$\begin{cases} u_{tt} + u_t - u_{xx} = 0 \text{ in } (0,\pi) \times (0,\infty) \\ u(x,0) = \sin x \text{ on } [0,\pi] + \times\{t=0\} \\ u_t(x,0) = 0 \text{ on } [0,\pi] \times \{t=0\} \\ u(0,t) = 0, \quad \{x=0\} \times (0,\infty) \\ u(\pi,t) = 0 \quad \{x=\pi\} \times (0,\infty) \end{cases} \tag{12}$$

14. Let $c > 0$ be such that $c^2 < 1$ and $f \in C^2([0,l])$. Then solve the following wave equation. [30]

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \text{ in } (0,1) \times (0,x) \\ u(x,x) = \sin x \text{ on } [0,1] + \times\{t=0\} \\ u_t(x,x) = 0 \text{ on } [0,1] \times \{t=0\} \\ u(0,t) = 0, \quad \{x=0\} \times (0,\infty) \\ u(1,t) = 0 \quad \{x=1\} \times (0,\infty) \end{cases} \tag{13}$$

15. Consider the following wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } (0,1) \times (0,1) \\ u(x,0) = 0 \text{ on } [0,1] + \times\{t=0\} \\ u_t(x,1) = 0 \text{ on } [0,1] \times \{t=1\} \\ u(0,t) = 0, \quad \{x=0\} \times (0,\infty) \\ u(1,t) = 0 \quad \{x=1\} \times (0,1) \end{cases} \tag{14}$$

(a) For each $k \in \mathbb{R}$ verify that $u_k(x,t) = k \sin \pi x \sin \pi t$ is a solution of [15]

(b) Find

$$\max_{(x,t)\in[0,1]\times[0,1]} |u_k(x,t)|$$

[10]

(c) Find

$$\min_{(x,t)\in[0,1]\times[0,1]} |u_k(x,t)|$$

[5]