भारतीय प्रौद्योगिकी संस्थान तिरुपति

IIT TIRUPATI

INDIAN INSTITUTE OF TECHNOLOGY TIRUPATI

DEPARTMENT OF MATHEMATICS AND STATISTICS

MA6024-PARTIAL DIFFERENTIAL EQUATIONS

Assignment-3

---

1. Let $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ and $u \in C^2(\overline{\Omega})$ be the solution of the Laplace equation

$$\begin{cases} \Delta u = 0 \ \text{ in } \ \Omega \end{cases} \tag{1}$$

If

$$\lim_{|(x,y)| \to \infty} u(x,y) = 0$$

Prove that

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|$$

[10]

2. Convert the following Laplace equation in polar coordinate

$$\begin{cases} u_{xx} + u_{yy} = 0 \ \text{ in } \ \Omega = \{(x,y) : x^2 + y^2 > a^2\} \\ u(r,\theta) = 1 + 3\sin\theta \ \text{ on } \ r = a \\ \lim_{r\to\infty} u(r,\theta) < \infty \end{cases} \tag{2}$$

Prove that $u(r,\theta) = 1 + \frac{3a}{r}\sin\theta$ [3]

3. Find the $\alpha, \beta, \gamma \in \mathbb{R}$ for which the following Laplace equation is not solvable

$$\begin{cases} u_{xx} + u_{yy} = 0 \ \text{ in } \ \Omega = \{(x,y) : x^2 + y^2 < 4\} \\ \frac{\partial u}{\partial \nu} = \alpha x^2 + \beta y + \gamma \ \text{ on } \ (x,y) \in \partial\Omega \end{cases} \tag{3}$$

[6]

4. Solve the following Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \ \text{ in } \ \Omega = \{(x,y) : x^2 + y^2 > 4\} \\ u(x,y) = y \ \text{ on } \ (x,y) \in \partial\Omega \\ \lim_{|x|+|y|\to\infty} u(x,y) = 0 \end{cases} \tag{4}$$

[8]

5. We say that $v \in C^2(\overline{\Omega})$ is subharmonic if

$$-\Delta v \leq 0 \quad \text{in} \quad \Omega$$

(a) Prove for subharmonic v that

$$v(\mathbf{x}) \leq \fint_{\partial B(\mathbf{x},\epsilon)} v(\mathbf{y}) dS(y) \quad \text{for all} \quad B(\mathbf{x}, r) \subset \Omega$$

[8]

(b) Prove that

$$\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$$

[5]

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove that v is harmonic. [5]

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic. [5]

6. Let $\Omega := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $u \in C^2(\Omega) \cap C^2(\overline{\Omega})$ be the solution of the Laplace equation.

$$\Delta u = 0 \quad \text{in} \quad \Omega \tag{5}$$

If u is bounded above in $\overline{\Omega}$ Prove that

$$\sup_{\overline{\Omega}} |u| = \sup_{\partial\Omega} |u|$$

[10]

7. Convert the following Laplace equation in polar coordinate

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega = \{(x, y) : x^2 + y^2 < a^2\} \\ u(r, \theta) = 1 + 3\sin\theta \quad \text{on} \quad r = a \end{cases} \tag{6}$$

Prove that $u(r, \theta) = 1 + \frac{3a}{r} \sin\theta$ [3]

8. Find the $\alpha, \beta, \gamma \in \mathbb{R}$ for which the following Laplace equation is solvable. If solvable fin the solution

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega = \{(x, y) : x^2 + y^2 < 4\} \\ \frac{\partial u}{\partial \nu} = \alpha x^2 + \beta y + \gamma \quad \text{on} \quad (x, y) \in \partial\Omega \end{cases} \tag{7}$$

[6+8]

9. Solve the following Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega = \{(x, y) : x^2 + y^2 > 9\} \\ u(x, y) = y \quad \text{on} \quad (x, y) \in \partial\Omega \\ \lim_{|x|+|y|\to\infty} u(x, y) = 0 \end{cases} \tag{8}$$

[8]

10. Prove that there exists a constant C depending only on n, such that

$$\max_{B(\mathbf{0},1)} |u| \leq (\max_{\partial B(\mathbf{0},1)} |f| + \max_{B(\mathbf{0},1)} |h|)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = h & \text{in } B^0(\mathbf{0},1) \\ u = f & \text{on } \partial B(\mathbf{0},1) \end{cases} \tag{9}$$

[15]

11. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $u \in C^2(\Omega) \cap C^2(\overline{\Omega})$ be the solution of the following equation

$$\Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0 \quad \text{in} \quad \Omega \tag{10}$$

Here, $a, b, c \in C^2(\overline{\Omega})$ and $c(x,y) < 0$ in Ω. Show that $u = 0$ on $\partial\Omega \implies u = 0$ in Ω

12. Convert the following Laplace equation in polar coordinate

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in} \quad \Omega = \{(x,y) : x^2 + y^2 < a^2\} \\ u(r,\theta) = 1 + 3\sin 2\theta & \text{on} \quad r = a \end{cases} \tag{11}$$

Find the value of $u(0,0)$. Is it possible to find the value of $u(0,0)$ without solving? [8+2]

13. Find the $\alpha, \beta, \gamma \in \mathbb{R}$ for which the following Laplace equation is solvable. If solvable fin the solution

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in} \quad \Omega = \{(x,y) : x^2 + y^2 < 16\} \\ \frac{\partial u}{\partial \nu} = \alpha x^2 + \beta y + \gamma & \text{on} \quad (x,y) \in \partial\Omega \end{cases} \tag{12}$$

[10]

14. Solve the following Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in} \quad \Omega = \{(x,y) : x^2 + y^2 > 9\} \\ u(x,y) = y & \text{on} \quad (x,y) \in \partial\Omega \\ \lim\limits_{|x|+|y|\to\infty} u(x,y) = 0 \end{cases} \tag{13}$$

[10]

15. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r - |\mathbf{x}|}{(r + |\mathbf{x}|)^{n-1}}u(0) \leq u(\mathbf{x}) \leq r^{n-2}\frac{r + |\mathbf{x}|}{(r - |\mathbf{x}|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^0(\mathbf{0}, r)$.

16. Let $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$ and u be a non-constant solution of the Laplace equation.

$$\Delta u = 0 \quad \text{in} \quad \Omega \tag{14}$$

Define $\mu : (0,a) \to \mathbb{R}$ by

$$\mu(r) = \max_{x^2+y^2=r^2} u(x,y)$$

Prove that $\frac{d\mu}{dr} \geq 0$ for all $r \in (0,a)$ [10]

17. Solve the following Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 1 \quad \text{in} \quad \Omega = \{(x,y) : x^2 + y^2 < a^2\} \\ u(x,y) = 0 \quad \text{on} \quad (x,y) \in \partial\Omega \end{cases} \tag{15}$$

[10]

18. Find the $\alpha, \beta, \gamma \in \mathbb{R}$ for which the following Laplace equation is not solvable

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega = \{(x,y) : x^2 + y^2 < 16\} \\ \frac{\partial u}{\partial \nu} = \alpha x^2 + \beta y + \gamma \quad \text{on} \quad (x,y) \in \partial\Omega \end{cases} \tag{16}$$

[10]

19. Solve the following Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega = \{(x,y) : x^2 + y^2 > 16\} \\ u(x,y) = y \quad \text{on} \quad (x,y) \in \partial\Omega \\ \lim_{|x|+|y| \to \infty} u(x,y) = 0 \end{cases} \tag{17}$$

[10]

20. Assume $g \in C(\partial B(\mathbf{0}, r))$ and define u by

$$u(\mathbf{x}) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int\limits_{\partial B(\mathbf{0},r} \frac{g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} dS(\mathbf{y}), \quad (x \in B^0(\mathbf{0}, r))$$

Then

(a) $u \in C^\infty(B^0(\mathbf{0}, r))$

(b) $\Delta u = 0$ in $B^0(\mathbf{0}, r)$

(c) $\lim\limits_{\substack{(\mathbf{x}) \to (\mathbf{x}^0) \\ \mathbf{x} \in B^0(\mathbf{0},r), t>0}} u(\mathbf{x}) = f(\mathbf{x}^0)$, for each point $\mathbf{x}^0 \in B^0(\mathbf{0}, r)$

21. Solve the following Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in} \quad (0,\pi) \times (0,\pi) \\ u(x,0) = 0 \quad \text{on} \quad [0,\pi] \times \{y = 0\} \\ u(x,\pi) = 0 \quad \text{on} \quad [0,\pi] \times \{y = \pi\} \\ u(0,y) = 0 \quad \text{on} \quad \{x = 0\} \times [0,\pi] \\ u(\pi,y) = \sin y \quad \text{on} \quad \{x = \pi\} \times [0,\pi] \end{cases} \tag{18}$$

Prove or disprove, there exists a point in $(p, q) \in (0, \pi) \times (0, \pi)$ such that $u(p, q) = 0$. [Bonus 5]