INDIAN INSTITUTE OF TECHNOLOGY TIRUPATI

Department of Mathematics and Statistics

MA6024 - Partial Differential Equations

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Mid-Semester

SEMESTER I — Duration: 90 Min

II MSc (Mathematics and Statistics) — Max. MARKS: 100

Part A

Answer all Questions $4 \times 5 = 20$

1. Classify the following PDEs as semilinear, quasilinear, fully nonlinear, linear, homogeneous and inhomogeneous.

   (a) $\sqrt{1+x^2y^2}u_{xyy} - \cos(xy^3)u_{xxy} + e^{-y^3}u_x - (5x^2 - 2xy + 3y^2)u = 0$

   Answer: Linear, Semilinear, Quasilinear, Homogeneous

   (b) $(y+u)u_x + yu_y = x - y$

   Answer: Quasilinear, inhomogeneous

2. Find the solution of the following PDE using method of characteristics.

$$u_x + 3y^{2/3}u_y = 2, \quad u(x,1) = 1 + x$$

Show that the solution $u(x,y)$ is not differentiable at $y = 0$.

Answer: Method 1

The initial curve is given by

$$\Gamma : \begin{cases} x = x_0(s) = x(s,0) = s \\ y = y_0(s) = y(s,0) = 1 \\ u = u_0(s) = u(s,0) = 1 + s \end{cases}$$

Characteristic system

$$C : \begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 3y^{2/3} \\ \frac{du}{dt} = 2 \end{cases}$$

Therefore, the characteristic curves are

$$\begin{cases} x(s,t) = t+s \\ y(s,t) = (t+1)^3 \\ u(s,t) = 1+s+2t \end{cases}$$

Hence $s = x+1-y^{1/3}, t = y^{1/3}-1$ and the solution is given by

$$u(x,y) = x + y^{1/3}$$

Since $u_y = \frac{1}{3}y^{-2/3}$, the partial derivative does not exist at $y = 0$ and hence it is not differentiable at $y = 0$.

Answer: Method 2

Characteristic system

$$\frac{dx}{1} = \frac{dy}{3y^{2/3}} = \frac{du}{2}$$

$$\frac{dx}{1} = \frac{dy}{3y^{2/3}} \implies x = y^{1/3} + C_1 \implies \phi(x,y) = x - y^{1/3}$$

$$\frac{du}{dx} = 2 \implies u - 2x = C_2 \implies \psi(x,y) = u - 2x$$

$$u - 2x = F(x - y^{1/3})$$

For given initial data $C_1 = x-1, C_2 = 1+x-2x = 1-x \implies C_1 = -C_2$

$$\implies \phi = -\psi \implies u - 2x = y^{1/3} - x \implies u = x + y^{1/3}$$

3. Show that

$$u_x + u_y + u = 1$$

with the initial curve

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = s + s^2 \\ u = u_0(s) = \sin s \end{cases}$$

has a unique solution.

Answer: This can be written as a quasilinear PDE $u_x + u_y = 1 - u$

$$C : \begin{cases} \frac{dx}{dt} = 1 = a(x,y,u) \\ \frac{dy}{dt} = 1 = b(x,y,u) \\ \frac{du}{dt} = 1 - u = c(x,y,u) \end{cases}$$

Obviously $a, b, c \in C^1(\Omega)$. Also, Γ is an initial smooth curve. Therefore, if we check the transversality condition,

$$T(s) \equiv \frac{dx_0}{ds} b(x_0(s), y_0(s), u_0(s)) - \frac{dy_0}{ds} b(x_0(s), y_0(s), u_0(s)) \neq 0, s \in [0,1],$$

then by existence and uniqueness theorem it has an unique solution. Now,

$$T(s) = 1.1 - (1+s).1 = -s \not\equiv 0$$

4. Show that all the characteristic curves of the PDE

$$(2x+u)u_x + (2y+u)u_y = u$$

through the point $(1,1)$ are given by the same straight line $x - y = 0$.

Answer:

Assume any arbitrary initial curve

$$\Gamma : \begin{cases} x = x_0(s) \\ y = y_0(s) \\ u = u_0(s) \end{cases}$$

Characteristic system

$$C : \begin{cases} \frac{dx}{dt} = 2x+u \\ \frac{dy}{dt} = 2y+u \\ \frac{du}{dt} = u \end{cases}$$

$$\frac{du}{dt} = u \implies \log u = t + C \implies u = Ce^t$$

$$\frac{dx}{dt} = 2x + Ce^t \implies x = C_1e^{2t} - Ce^t$$

$$\frac{dy}{dt} = 2y + Ce^t \implies y = C_2e^{2t} - Ce^t$$

Therefore, the characteristic curves are

$$\begin{cases} x(s,t) = x_0(s)e^{2t} + u_0(s)(e^{2t} - e^t) \\ y(s,t) = y_0(s)e^{2t} + u_0(s)(e^{2t} - e^t) \\ u(s,t) = u_0(s)e^t \end{cases}$$

If the curve passes through the point $(1,1)$, then from the characteristic curves, it is clear that characteristic curves are the same straight line $x - y = 0$. Will there exists a characteristic curve passing through $(1,0)$?

Part B

Answer any Four Questions $4 \times 15 = 60$

1. Show that the solution of Burger's equation for the following initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x \le 0 \\ 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x \ge 1 \end{cases}$$

is given by

$$u(x,y) = \begin{cases} 1 & \text{if } x \le y \\ \dfrac{1-x}{1-y} & \text{if } y \le x \le 1 \\ 0 & \text{if } x \ge 1 \end{cases}$$

Answer: The Burger's equation is given by

$$u_t + uu_x = 0$$

Suppose $u(x,0) = F(x)$ is the initial curve, then The characteristic curve satisfies

$$\frac{dx}{dt} = u(x(t),t) = u(x(0),0) = F(x(0))$$

Upon integrating, we obtain

$$x(t) = u(x(0),0)t + x(0) = F(x(0))t + x(0)$$

The solution of the IVP is given by

$$u(x,t) = F(x-ut)$$

$$F(x) = \begin{cases} u_l & \text{if } x \le 0 \\ u_l - ax & \text{if } 0 \le x \le x_r \\ u_r & \text{if } x \ge x_r \end{cases}$$

The characteristic curves are given by

$$x(t) = \begin{cases} u_l t + x(0) & \text{if } x(0) \le 0 \\ (u_l - ax(0))t + x(0) & \text{if } 0 \le x(0) \le x_r \\ u_r t + x(0) & \text{if } x(0) \ge x_r \end{cases}$$

Solving for t we have

$$t = \begin{cases} \frac{x-x(0)}{u_l} & \text{if } x(0) \le 0 \\ \frac{x-x(0)}{u_l - ax(0)} & \text{if } 0 \le x(0) \le x_r \\ \frac{x-x(0)}{u_r} & \text{if } x(0) \ge x_r \end{cases}$$

The solution is given by

$$u(x,t) = F(x-ut) = \begin{cases} u_l & \text{if } x - ut \le 0 \\ u_l - a(x-ut) & \text{if } 0 \le x - ut \le x_r \\ u_r & \text{if } x - ut \ge x_r \end{cases}$$

or For $x - u_l t \le 0$, we have $u = u_l$ For $x - u_r t \ge 0$, we have $u = u_r$ For $0 \le x - ut \le x_r$, we have

$$u = u_l - a(x-ut) = u_l - ax + aut \implies u(1-at) = u_l - ax \implies u = \frac{u_l - ax}{1-at}$$

$$u(x,t) = \begin{cases} u_l & \text{if } x \leq ut \\ \frac{u_l - ax}{1-at} & \text{if } u_l t \leq x \leq x_r + u_r t \\ u_r & \text{if } x >\geq x_r + u_r t \end{cases}$$

For our problem, $a = 1, u_1 = 1, u_r = 0, x_l = 0, x_r = 1$ Hence,

$$u(x,t) = \begin{cases} 1 & \text{if } x \leq t \\ \frac{1-x}{1-t} & \text{if } t \leq x \leq 1 \\ 0 & \text{if } x >\geq 1 \end{cases}$$

By replacing t by y, we can obtain the solution for the Burger's equation $u_y + uu_x = 0$.

2. Derive d'Alembert's formula using a change of variables [8+4+3]

   (a) Show that general solution of the PDE $u_{xy} = 0$ is given by

   $$u(x,y) = F(x) + G(y)$$

   for differentiable functions F and G

   Answer: Suppose $u(x,y) = F(x) + G(y)$, then $u_x = F'(x) = f(x) \implies u_{xy} = 0$ Conversely suppose $u_{xy} = 0$, then let $v = u_x \implies v_y = 0 \implies v = f(x)$. Let F be such that $f(x) = F'(x)$. Then $u_x = F'(x)$. Integrating it w.r.t x, we get $u = F(x) + G(y)$ for some G.

   (b) Using the change of variables, $\xi = x + t, \eta = x - t$, show that $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

   Answer: $\xi_x = x, \xi_t = 1, \eta_x = 1, \eta_t = -1$. Now,

   $$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$

   $$u_t = u_\xi \xi_t + u_\eta \eta_t = u_\xi - u_\eta$$

   $$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

   $$u_{tt} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

   $$u_{tt} - u_{xx} = -4u_{\xi\eta}$$

   Hence $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

   (c) Use (a) and (b) to derive d'Alembert's formula.

   Answer: Let us solve the following wave equation

   $$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = f & \text{on } \mathbb{R} \times \{t = 0\} \\ u_t = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

   By Part (b), we have $u_{\xi\eta} = 0$. By part (a), we have $u(\xi, \eta) = F(\xi) + G(\eta)$

   $$u(x,t) = F(x+t) + G(x-t)$$

$$u_t = F'(x+t) - G'(x-t)$$

Since $u(x,0) = f$, we have

$$u(x,0) = F(x) + G(x) = f(x)$$

Since $u_t(x,0) = g$, we have

$$u_t(x,0) = F'(x) + G'(x) = g(x)$$

Let $H(x) = F(x) + G(x)$, then $H'(x) = g(x)$ Hence

$$F(x) = \frac{1}{2}(f(x) + H(x)), \quad G(x) = \frac{1}{2}(f(x) - H(x))$$

Hence

$$\begin{aligned} u(x,t) &= F(x+t) + G(x-t) \\ &= \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}H(x+t) - H(x-t) \\ &= \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int\limits_{x-t}^{x+t} h(y)dy \end{aligned}$$

Hence the formula.

3. We say that a function is subharmonic if $\Delta u \geq 0$. In particular, every harmonic function is subharmonic. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and suppose that u is a subharmonic function on Ω. Assume that u extends to a continuous function on $\overline{\Omega}$.

   (a) Show that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

Answer:

For a given $\epsilon > 0$, define

$$u^\epsilon(\mathbf{x}) := u(\mathbf{x}) + \epsilon|\mathbf{x}|^2$$

Since u extends to a continuous function on $\overline{\Omega}$, so does u^ϵ. Now,

$$\Delta u^\epsilon = \Delta u + \Delta(\epsilon|\mathbf{x}|^2) = \Delta y + 2\epsilon n \geq 2\epsilon n > 0$$

Since Ω is bounded and u^ϵ is continuous on $\overline{\Omega}$, it attains its maximum. Further, it claims that u^ϵ attains its maximum on $\partial\Omega$. Let x_M^ϵ and x_M be respectively element at which u^ϵ and u attain their maximum on $\partial\Omega$. Then

$$u(x_M) = \max_{y\in\partial\Omega} u(y) \quad \text{and} \quad u^\epsilon(x_M^\epsilon) = \max_{y\in\partial\Omega} u^\epsilon(y)$$

Let $x \in \Omega$, in order to prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

it is sufficient to prove that

$$u(x) \le u(x_M)$$

Obviously, for all $\epsilon > 0$

$$u^\epsilon(x) \le u^\epsilon(x_M^\epsilon)$$

Hence

$$u(x) = u^\epsilon(x) - \epsilon|\mathbf{x}|^2 \le u^\epsilon(x_M^\epsilon) - \epsilon|\mathbf{x}|^2 = u(x_M^\epsilon) + \epsilon|\mathbf{x}_M^\epsilon|^2 - \epsilon|\mathbf{x}|^2$$

Since $x_M^\epsilon \in \partial\Omega$, we have

$$u(x_M^\epsilon) \le u(x_M)$$

Since Ω is bounded, there exist an $M > 0$ such that $|\mathbf{y}| \le M$ for all $y \in \overline{\Omega}$ Hence

$$u(x) \le u(x_M) + \epsilon(L^2 - |\mathbf{x}|^2)$$

As $\epsilon \to 0$, we get

$$u(x) \le u(x_M)$$

(b) For $n = 2$ and the function $u(x, y) = x^2 + y^2$ on the closed unit ball

$$B(0, 1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$$

calculate Δu and deduce that u is subharmonic.

Answer:

Since $u(x, y) = x^2 + y^2$, we have $\Delta u = 4 > 0$. Therefore, u is subharmonic.

[10+5]

4. Suppose that $u : \mathbb{R}^3 \to \mathbb{R}$ is a harmonic function. [10+5]

(a) Show that for all $x \in \mathbb{R}^3$ and for all $r > 0$,

$$u(x) = \frac{3}{4\pi r^3} \int\limits_{B(\mathbf{x},r)} u(y)dy$$

Hint: Use Mean Value Property From Mean value property we have

$$\phi(r) = \lim_{t\to 0} \phi(t) = \lim_{t\to 0} \fint\limits_{\partial B(\mathbf{x},t)} u(\mathbf{y})dS(\mathbf{y}) = u(\mathbf{x})$$

From Coarea formula, we have

$$\int\limits_{B(\mathbf{x},r)} f d\mathbf{x} = \int_0^r \left(\int\limits_{\partial B(\mathbf{x},s)} f dS \right) ds$$

$$\fint_{B(\mathbf{x},r)} u dy = \frac{3}{4\pi r^3} \int_{B(\mathbf{x},r)} u dy = \frac{3}{4\pi r^3} \int_0^r \left(\int_{\partial B(\mathbf{x},s)} u dS \right) ds$$
$$= \frac{3}{4\pi r^3} \int_0^r 4\pi s^2 u(\mathbf{x}) ds = u(\mathbf{x})$$

Hence

$$u(x) = \frac{3}{4\pi r^3} \int_{B(\mathbf{x},r)} u(y) dy$$

(b) Suppose

$$\int_{\mathbb{R}^3} |u(y)| dy < \infty,$$

show that

$$u(x) = 0 \quad \forall x \in \mathbb{R}^3$$

Answer: From the above part (a), we have proved that

$$u(x) = \frac{3}{4\pi r^3} \int_{B(\mathbf{x},r)} u(y) dy$$

Therefore

$$|u(x)| = \left| \frac{3}{4\pi r^3} \int_{B(\mathbf{x},r)} u(y) dy \right| \leq \frac{3}{4\pi r^3} \int_{B(\mathbf{x},r)} |u(y)| dy \leq \int_{\mathbb{R}^3} |u(y)| dy$$

To cover $\mathbb{R}^3$, we can let $r \to \infty$ and hence $|u(x)| \leq 0$ and hence $u(x) = 0$. Since x is arbitrary, the proof follows.

5. Fix $\mathbf{x} \in \mathbb{R}^n$ and let u satisfy the following wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u = f \text{ on } \mathbb{R}^n \times \{t = 0\} \\ u_t = g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Let $U(\mathbf{x}; r, t), F(\mathbf{x}; r, t)$ and $G(\mathbf{x}; r, t)$ be respectively the average of u, f and g over the sphere $\partial B(\mathbf{x}, r)$. Prove that $U \in C^m(\overline{\mathbb{R}}_+ \times [0, \infty])$ and U satisfies the following Euler-Poisson-Darboux equation

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ U = F \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ U_t = G \text{ on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

Answer:

Let $x \in \mathbb{R}^n, t > 0, r > 0$. Define

$$U(x;r,t) := \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y},t)dS(\mathbf{y}) \tag{1}$$

the average of $u(\mathbf{x},t)$ over the sphere $\partial B(\mathbf{x},r)$. Similarly

$$\begin{cases} F(x;r,t) := \fint_{\partial B(\mathbf{x},r)} f(\mathbf{y},t)dS(\mathbf{y}) \\ G(x;r,t) := \fint_{\partial B(\mathbf{x},r)} g(\mathbf{y},t)dS(\mathbf{y}) \end{cases} \tag{2}$$

(1) can also be written as

$$U(x;r,t) = \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y},t)dS(\mathbf{y}) = \fint_{\partial B(\mathbf{0},1)} u(\mathbf{x}+r\mathbf{z},t)dS(\mathbf{z})$$

Differentiating it w.r.t r, we obtain the following:

$$\begin{aligned} U_r(x;r,t) &= \fint_{\partial B(\mathbf{0},1)} Du(\mathbf{x}+r\mathbf{z},t).\mathbf{z}dS(\mathbf{z}) \\ &= \fint_{\partial B(\mathbf{x},r)} Du(\mathbf{y},t).\frac{\mathbf{y}-\mathbf{x}}{r}dS(\mathbf{y}) \\ &= \fint_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu}dS(\mathbf{y}) = \frac{r}{n}\fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y} \end{aligned}$$

Since

$$U_r(x;r,t) = \frac{r}{n}\fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y} \implies \lim_{r\to 0^+} U_r(x;r,t) = 0$$

Also, we can deduce that

$$\lim_{r\to 0^+} U_{rr}(x;r,t) = \frac{1}{n}\Delta u(x,t)$$

By computing through U_{rrr}, U_{rrrr}, etc., we can obtain that $U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty])$ Now,

$$U_r(x;r,t) = \frac{r}{n}\fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y} = \frac{r}{n}\fint_{B(\mathbf{x},r)} u_{tt}d\mathbf{y}$$

$$U_r(x;r,t) = \frac{r}{n}\frac{1}{\alpha(n)r^n}\int_{B(\mathbf{x},r)} u_{tt}d\mathbf{y} = \frac{1}{n\alpha(n)r^{n-1}}\int_{B(\mathbf{x},r)} u_{tt}d\mathbf{y}$$

$$\implies r^{n-1}U_r = \frac{1}{n\alpha(n)}\int_{B(\mathbf{x},r)} u_{tt}d\mathbf{y}$$

Now differentiating w.r.t r, we obtain

$$(r^{n-1}U_r)_r = \frac{1}{n\alpha(n)} \int\limits_{\partial B(\mathbf{x},r)} u_{tt} dS$$

$$(n-1)r^{n-2}U_r + r^{n-1}U_r r = r^{n-1}\left(\frac{1}{n\alpha(n)r^{n-1}} \int\limits_{\partial B(\mathbf{x},r)} u_{tt} dS \right)$$

$$(n-1)r^{n-2}U_r + r^{n-1}U_r = r^{n-1} \fint\limits_{\partial B(\mathbf{x},r)} u_{tt} = r^{n-1}U_{tt}$$

Hence

$$U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0$$

Part C

Answer any One Question $1 \times 20 = 20$

1. Solve the following boundary value problem using method of separation of variables (Discuss all cases)

$$\begin{cases} u_t - u_{xx} = \sin(2\pi x) + \sin(3\pi x), & \text{in} \quad (0,1) \times (0,\infty) \\ u(x,0) = 0, & \text{on} \quad [0,1] \times \{t=0\} \\ u(0,t) = 0, & \text{on} \quad \{x=0\} \times (0,\infty) \\ u(1,t) = 0, & \text{on} \quad \{x=1\} \times (0,\infty) \end{cases}$$

Answer: First, let us solve the following general inhomogeneous equation

$$\begin{cases} u_t - u_{xx} = h(x,t), & \text{in} \quad (0,1) \times (0,\infty) \\ u(x,0) = f(x), & \text{on} \quad [0,1] \times \{t=0\} \\ u(0,t) = 0, & \text{on} \quad \{x=0\} \times (0,\infty) \\ u(1,t) = 0, & \text{on} \quad \{x=1\} \times (0,\infty) \end{cases}$$

Let $u(x,t) = X(x)T(t)$, then we obtain

$$u_t = X(x)T'(t) \qquad u_{xx} = X''(x)T(t)$$

We obtain that

$$X(x)T'(t) = X''(x)T(t) \implies \frac{T'}{T} = \frac{X''}{X} = \mu(constant)$$

Therefore, we have

$$\frac{dT}{dt} - \mu T = 0, \qquad \frac{d^2X}{dx^2} - \mu X = 0$$

where μ is any real constant.

Let us look at the boundary conditions.

$$u(0,t) = 0 \implies X(0)T(t) = 0$$

If $T \equiv 0$, then $u(x,t) \equiv 0$ which is a trivial solution and we are not interested on this solution. Therefore, $T(t) \not\equiv 0$, consequently, $X(0) = 0$. Similarly,

$$u(1,t) = 0 \implies X(1)T(t) = 0$$

If $T \equiv 0$, then $u(x,t) \equiv 0$ which is a trivial solution and we are not interested on this solution. Therefore, $T(t) \not\equiv 0$, consequently, $X(1) = 0$.

So, let us consider the following problem,

$$\begin{aligned} X'' - \mu X = 0, x \in [0,1] \\ X(0) = 0 \\ X(1) = 0 \end{aligned}$$

Now, let us consider three cases, $\mu > 0, = 0, < 0$.
Case 1: $\mu = 0$
Then

$$X'' = 0 \implies X' = a \implies X = ax + b$$
$$X(0) = 0 \implies b = 0$$
$$X(1) = 0 \implies a = 0, 1 \neq 0 \implies a = 0$$
$$X(x) \equiv 0 \implies u(x,t) = X(x)T(t) = 0 \implies u \equiv 0$$

We are not interested in the solution $u \equiv 0$
Case 2: $\mu > 0$**, let** $\mu = \nu^2$
Then

$$X'' - \nu^2 X \implies m^2 - \nu^2 = 0 \quad (\text{ auxiliary equation })$$
$$\implies m = \pm\nu \implies X(x) = Ae^{\nu x} + Be^{-\nu x}$$
$$X(0) = 0 \implies A + B = 0 \implies A = -B \implies X(x) = A(e^{\nu x} - e^{-\nu x})$$
$$X(1) = 0 \implies A(e^{\nu} - e^{-\nu}) = 0$$
$$\implies e^{\nu} = e^{-\nu} \quad \text{or} \quad A = 0$$
$$\implies 1 = 0 \quad \text{or} \quad \nu = 0 \quad \text{or} \quad A = 0$$
$$1 \neq 0, \nu = 0 \implies \mu = 0 \quad (\text{Case 1, discussed}) \implies A = 0$$
$$X(x) \equiv 0 \implies u(x,t) = X(x)T(t) = 0 \implies u \equiv 0$$

Again, We are not interested in the solution $u \equiv 0$
Case 3: $\mu < 0$**, let** $\mu = -\nu^2$
Then

$$X'' + \nu^2 X \implies m^2 + \nu^2 = 0 \quad (\text{ auxiliary equation })$$

$$\implies m = \pm \nu i \implies X(x) = A\cos(\nu x) + B\sin(\nu x)$$

$$X(0) = 0 \implies A + 0 = 0 \implies A = 0 \implies X(x) = B\sin(\nu x)$$

$$X(1) = 0 \implies B\sin(\nu) = 0 \implies \nu_n = n\pi, n \in \mathbb{Z}$$

Let

$$X_n(x) = B_n \sin(n\pi x), n \in \mathbb{Z}$$

Observe that

$$X_n(x) = -B_n \sin(|n|\pi x), n < 0$$

$$\implies X_n(x) = C_n \sin(n\pi x), n > 0$$

and we are not interested in

$$X_n(x) = 0, n = 0$$

Therefore,

$$X_n(x) = B_n \sin(n\pi x), n \in \mathbb{N}$$

Therefore, we have

$$u_n(x,t) = X_n(x)T_n(t), n \in \mathbb{N}$$

Now, assume the following as the solution of the general inhomogeneous equation

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x) \tag{3}$$

For the inhomogeneous equation, we have

$$h(x,t) = u_t - u_{xx} = \sum_{n=1}^{\infty}[T_n'(t) + n^2\pi^2 T_n(t)]\sin(n\pi x)$$

If we expand $h(x,t)$ and $f(x)$ using Fourier Series, we have

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t)\sin(n\pi x)$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x)$$

where

$$h_n(t) = \int_0^1 h(x,t)\sin(n\pi x)\,dx, \quad n \in \mathbb{N}$$

$$f_n = \int_0^1 f(x)\sin(n\pi x)\,dx, \quad n \in \mathbb{N}$$

Therefore, we have

$$\sum_{n=1}^{\infty} T_n(t)\sin(n\pi x) = \sum_{n=1}^{\infty} h_n(t)\sin(n\pi x)$$

Hence

$$T_n'(t) + n^2\pi^2 T_n(t) = h_n(t)$$

Further,

$$T_n(0) = f_n$$

Hence,

$$T_n(t) = f_n e^{-n^2\pi^2 t} + \int_0^1 e^{-n^2\pi^2(t-s)} h_n(s) ds$$

In our problem,

$$f(x) = 0 \implies f_n = 0$$

and

$$h(x,t) = \sin(2\pi x) + \sin(3\pi x) \implies h_2(t) = 1, h_3(t) = 1, h_n(t) = 0, n \in \mathbb{N} \setminus \{2,3\}$$

Hence

$$T_2(t) = \int_0^1 e^{-4\pi^2(t-s)} ds = \frac{1}{4\pi^2} e^{-4\pi^2 t} [e^{4\pi^2 s}]_0^1 = \frac{1}{4\pi^2}[e^{4\pi^2 t} - 1]$$

, similarly

$$T_3(t) = \int_0^1 e^{-9\pi^2(t-s)} ds = \frac{1}{9\pi^2}[e^{9\pi^2 t} - 1]$$

Therefore,

$$u(x,t) = \frac{1}{4\pi^2}[e^{4\pi^2 t} - 1] \sin(2\pi x) + \frac{1}{9\pi^2}[e^{9\pi^2 t} - 1] \sin(3\pi x)$$

2. Let u solves the following wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in} \quad \mathbb{R}^3 \times (0,\infty) \\ u = f, & \text{on} \quad \mathbb{R}^3 \times \{t = 0\} \\ u_t = g, & \text{on} \quad \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Suppose that $f, g \in C^\infty(\mathbb{R}^3)$ have compact support. Show that there exists a constant $C > 0$ such that

$$|u(x,t)| \leq \frac{C}{t}, \quad \forall x \in \mathbb{R}^3, t > 0$$

Hint: Use Kirchhoff's formula.

Answer: Since $f, g \in C^\infty(\mathbb{R}^3)$ have a compact support, $support f \subset B(\mathbf{0}, r)$ and $support g \subset B(\mathbf{0}, r)$. Also, there exists M such that $f(\mathbf{x}) \leq M, |Df(\mathbf{x})| \leq M, g(\mathbf{x}) \leq M$. By Kirchhoff's formula we have that

$$u(\mathbf{x}, t) = \fint_{\partial B(\mathbf{x},t)} tg(\mathbf{y}) + f(\mathbf{y}) + Df(\mathbf{y}).(\mathbf{y} - \mathbf{x}) dS(\mathbf{y}), (\mathbf{x} \in \mathbb{R}^3, t > 0)$$

Let $\Omega = \partial B(\mathbf{x}, t) \cap B(\mathbf{0}, r)$. Then

$$\text{Area of } \Omega \leq \text{Area of } \partial B(\mathbf{0}, r)$$

For $t > 0$, we have

$$\left| \fint_{\partial B(\mathbf{x},t)} tg(\mathbf{y}) + Df(\mathbf{y}).(\mathbf{y}-\mathbf{x})dS(\mathbf{y}) \right| = \frac{1}{4\pi t^2} \left| \fint_{\partial B(\mathbf{x},t)} tg(\mathbf{y}) + Df(\mathbf{y}).(\mathbf{y}-\mathbf{x})dS(\mathbf{y}) \right|$$
$$\leq \frac{1}{4\pi t^2} \left| \fint_{\Omega} tg(\mathbf{y}) + Df(\mathbf{y}).(\mathbf{y}-\mathbf{x})dS(\mathbf{y}) \right|$$
$$\leq \frac{1}{4\pi t^2} \fint_{\Omega} t|g(\mathbf{y})| + |Df(\mathbf{y})|.|(\mathbf{y}-\mathbf{x})|dS(\mathbf{y})$$
$$\leq \frac{4\pi t^2}{4\pi r^2}(tM + tM)$$
$$= \frac{2r^2 M}{t}$$

For $t > 1$, we have

$$\text{Area of } \Omega \leq \text{Area of } \partial B(\mathbf{x}, t)$$

$$\left| \fint_{\partial B(\mathbf{x},t)} g(\mathbf{y}) \right| = \frac{1}{4\pi t^2} \left| \fint_{\partial B(\mathbf{x},t)} g(\mathbf{y}) \right|$$
$$\leq \frac{1}{4\pi t^2} \fint_{\Omega} |g(\mathbf{y})| dS(\mathbf{y})$$
$$\leq \frac{4\pi t^2}{4\pi r^2} M$$
$$= \frac{r^2 M}{t^2} \leq \frac{r^2 M}{t}$$

For $0 \leq t \leq 1$, we have

$$\left| \fint_{\partial B(\mathbf{x},t)} g(\mathbf{y}) \right| = \frac{1}{4\pi t^2} \left| \fint_{\partial B(\mathbf{x},t)} g(\mathbf{y}) \right|$$
$$\leq \frac{1}{4\pi t^2} \fint_{\Omega} |g(\mathbf{y})| dS(\mathbf{y})$$
$$\leq \frac{4\pi t^2}{4\pi t^2} M$$
$$\leq \frac{M}{t}$$

Therefore, if we choose $r > 1$, we have

$$|u(\mathbf{x},t)| = \left| \fint_{\partial B(\mathbf{x},t)} tg(\mathbf{y}) + f(\mathbf{y}) + Df(\mathbf{y}).(\mathbf{y}-\mathbf{x})dS(\mathbf{y}) \right| \leq \frac{2r^2 M + r^2 M + M}{t} \leq \frac{C}{t}$$

All the best
