भारतीय प्रौद्योगिकी संस्थान तिरुपति

INDIAN INSTITUTE OF TECHNOLOGY TIRUPATI
DEPARTMENT OF MATHEMATICS AND STATISTICS

MA6024-PARTIAL DIFFERENTIAL EQUATIONS

Assignment-6

For each questions provide justification and theorems (wherever possible)

1. Consider the PDE
$$P(x,y)\frac{\partial^2 u}{\partial x^2} + e^{x^2}e^{y^2}\frac{\partial^2 u}{\partial x \partial y} + Q(x,y)\frac{\partial^2 u}{\partial y^2} + e^{2x}\frac{\partial u}{\partial x} + e^{y}\frac{\partial u}{\partial y} = 0$$
where P and Q are polynomials in two variables with real coefficients. Then which of the following is true for all choices of P and Q?

   (a) There exists $R > 0$ such that the PDE is elliptic in $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > R\}$
   (b) There exists $R > 0$ such that the PDE is hyperbolic in $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > R\}$
   (c) There exists $R > 0$ such that the PDE is parabolic in $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > R\}$
   (d) There exists $R > 0$ such that the PDE is hyperbolic in $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R\}$

2. Let u be the unique solution of the following heat equation
$$\begin{cases} u_t = u_{xx} & (x,t) \in (0,1) \times (0,\infty) \\ u(x,0) = \sin \pi x & x \in (0,1) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(1,t) = 0 & t \in (0,\infty) \end{cases}$$
Then which of the following is true?

   (a) There exists $(x,t) \in (0,1) \times (0,\infty)$ such that $u(x,t) = 0$
   (b) There exists $(x,t) \in (0,1) \times (0,\infty)$ such that $u_t(x,t) = 0$
   (c) The function $e^t u(x,t)$ is bounded for $(x,t) \in (0,1) \times (0,\infty)$
   (d) There exists $(x,t) \in (0,1) \times (0,\infty)$ such that $u(x,t) > 1$

3. Let u be the solution of the following Laplace equation
$$\begin{cases} u_{xx} = u_{yy} & (x,y) \in (0,\pi) \times (0,\pi) \\ u(x,0) = 0 & x \in (0,\pi) \\ u(x,\pi) = \sin(2x) & x \in (0,\pi) \\ u(0,y) = 0 & y \in (0,\pi) \\ u(\pi,y) = 0 & y \in (0,\pi) \end{cases}$$
Then which of the following is true?

(a) $\max_{x,y\in[0,\pi]} u(x,y) = 1$

(b) $u(x_0, y_0) = 1$ for some $(x_0, y_0) \in (0,\pi) \times (0,\pi)$

(c) $u(x,y) > -1$ for all $(x,y) \in (0,\pi) \times (0,\pi)$

(d) $\max_{x,y\in[0,\pi]} u(x,y) > -1$

4. Let u be the solution of the following Heat equation

$$\begin{cases} u_{tt} = u_{xx} & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

where f, g are in $C^2(\mathbb{R})$ and satisfy the following conditions

(i) $f(x) = g(x) = 0$ for $x \leq 0$

(ii) $0 < f(x) \leq 1$ for $x > 0$

(iii) $g(x) > 0$ for $x > 0$

(iv) $\int_0^\infty g(x)dx < \infty$

Then which of the following statements are true?

(a) $u(x,t) = 0$ for all $x \leq 0$ and $t > 0$

(b) u is bounded on $\mathbb{R} \times (0,\infty)$

(c) $u(x,t) = 0$ whenever $x + t < 0$

(d) $u(x,t) = 0$, for some (x,t) satisfying $x + t > 0$

5. Let u be the solution of the following PDE

$$\begin{cases} u_{tt} - u_{xx} = e^x + 6t & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = \sin(x) & x \in \mathbb{R} \\ u_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

Then the value of $u\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is _____

(a) $e^{\pi/2}\left(1 + \frac{1}{2}e^{\pi/2}\right) + \left(\frac{\pi^2+4}{8}\right)$

(b) $e^{\pi/2}\left(1 + \frac{1}{2}e^{\pi/2}\right) + \left(\frac{\pi^2-4}{8}\right)$

(c) $e^{\pi/2}\left(1 - \frac{1}{2}e^{\pi/2}\right) - \left(\frac{\pi^2+4}{8}\right)$

(d) $e^{\pi/2}\left(1 - \frac{1}{2}e^{\pi/2}\right) - \left(\frac{\pi^2-4}{8}\right)$

6. Let u be the solution of the following heat equation

$$\begin{cases} u_t = u_{xx} & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{elsewhere} \end{cases} \end{cases}$$

Then the value of $\lim_{t \to 0} u(1,t) =$ _____

(a) e

(b) π

(c) $\frac{1}{2}$

(d) 1

7. Consider the following IVP

$$xu_x + tu_t = u + 1, x \in \mathbb{R}, t \geq 0$$

$u(x,t) = x^2, t = x^2$. Then

(a) The solution is singular at $(0,0)$

(b) The given space curve $(x,t,u) = (\xi, \xi^2, \xi^2)$ is not a characteristic curve at $(0,0)$

(c) There is no base-characteristic curve in the (x,t) plane passing through $(0,0)$

(d) A necessary condition for the IVP to have a unique C^1 solution at $(0,0)$ does not hold.

8. Consider the following Burger's equation

$$u_t + uu_x = 1, x \in \mathbb{R}, t > 0$$

and the initial condition $u\left(\frac{t^2}{4}, t\right) = \frac{t}{2}$. Then the IVP has

(a) Only one solution

(b) Two solutions

(c) An infinite number of solutions

(d) Solutions none of which is differentiable on the characteristic base curve

9. The Cauchy problem

$$2u_x + 3u_y = 5$$

$u = 1$ on the line $3x - 2y = 0$ has

(a) Exactly one solution

(b) Exactly two solutions

(c) Infinitely many solutions

(d) No solution

10. Let u be the solution of the following PDE

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

where f satisfies the following relations $f(x) = x(1-x)$ for all $x \in [0,1]$ and $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Then $u\left(\frac{1}{2}, \frac{5}{4}\right)$ is ____

(a) $\frac{1}{8}$
(b) $\frac{1}{16}$
(c) $\frac{3}{16}$
(d) $\frac{5}{16}$

11. Let u be the unique solution of the following heat equation

$$\begin{cases} u_t = u_{xx} & (x,t) \in (0,1) \times (0,\infty) \\ u(x,0) = 1 + x + \sin(\pi x)\cos(\pi x) & x \in (0,1) \\ u(0,t) = 1 & t \in (0,\infty) \\ u(1,t) = 2 & t \in (0,\infty) \end{cases}$$

Then

(a) $u\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{3}{2}$
(b) $u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{3}{2}$
(c) $u\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{5}{4} + \frac{1}{2}e^{-3\pi^2}$
(d) $u\left(\frac{1}{4}, 1\right) = \frac{5}{4} + \frac{1}{2}e^{-4\pi^2}$

12. Let a be a fixed constant. Consider the first order partial differential equation

$$u_t + a u_x = 0, x \in \mathbb{R}, t > 0$$

with the initial data $u(x,0) = u_0(x), x \in \mathbb{R}$ where u_0 is a continuously differentiable function. Consider the following two statements. S_1 : There exists a bounded function u_0 for which the solution u is bounded S_2 : If u_0 vanishes outside a compact set then for each fixed $T > 0$ there exists a compact set $K_T \subset \mathbb{R}$ such that $u(x,t)$ vanishes for $x \notin K_T$. Which of the following are true

(a) S_1 is true and S_2 is false
(b) Both S_1 and S_2 are true
(c) S_1 is false and S_2 is true

(d) Both S_1 and S_2 are false

13. Let D denote the unit disc given by $\{(x,y) : x^2 + y^2 < 1\}$ and let D^c be its complement in the plane. The PDE

$$(x^2 - 1)u_{xx} + 2yu_{xy} - u_{yy} = 0$$

is

(a) Parabolic for all $(x,y) \in D^c$

(b) Parabolic for all $(x,y) \in D$

(c) hyperbolic for all $(x,y) \in D^c$

(d) hyperbolic for all $(x,y) \in D$

14. Let u be the solution of the following PDE

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = x^3 & x \in \mathbb{R} \\ u_t(x,0) = \sin(x) & x \in \mathbb{R} \end{cases}$$

Then the value of $u(\pi,\pi)$ is _____

(a) $4\pi^3$

(b) π^3

(c) 0

(d) 4

15. Let u be the solution of the following PDE

$$\begin{cases} u_t - u_{xx} = 0 & (x,t) \in [0,\pi] \times [0,T] \\ u(x,0) = \phi(x) & x \in [0,\pi] \\ u(0,t) = 0 & t \in [0,T] \\ u(\pi,t) = 0 & t \in [0,T] \end{cases}$$

If $f(x) = u(x,T)$, then which following is true for a suitable kernel $k(x,y)$?

(a) $\int_0^\pi k(x,y)\phi(y)dy = f(x), \quad 0 \le x \le \pi$

(b) $\phi(x) + \int_0^\pi k(x,y)\phi(y)dy = f(x), \quad 0 \le x \le \pi$

(c) $\int_0^x k(x,y)\phi(y)dy = f(x), \quad 0 \le x \le \pi$

(d) $\phi(x) + \int\limits_0^x k(x,y)\phi(y)dy = f(x), \quad 0 \le x \le \pi$

16. Consider the second order PDE

$$8u_{xx} - 2u_{xy} - 3u_{yy} = 0$$

Then which of the following are correct?

(a) The equation is elliptic

(b) The equation is hyperbolic

(c) The general solution is $u = f\left(y - \frac{x}{2}\right) + g\left(y + \frac{3x}{4}\right)$ for arbitrary differentiable functions f and g.

(d) The general solution is $u = f\left(y + \frac{x}{2}\right) + g\left(y - \frac{3x}{4}\right)$ for arbitrary differentiable functions f and g.

17. The solution of the following PDE

$$u_t - xu_x + 1 - u = 0, x \in \mathbb{R}, t > 0$$

subject to $u(x,0) = g(x)$ is

(a) $u(x,t) = 1 - e^{-t}(1 - g(xe^t))$

(b) $u(x,t) = 1 + e^{-t}(1 - g(xe^t))$

(c) $u(x,t) = 1 - e^{-t}(1 - g(xe^{-t}))$

(d) $u(x,t) = e^{-t}(1 - g(xe^t))$

18. Suppose $u \in C^2(B)$, B is the unit ball in $\mathbb{R}^2$ satisfies

$$\Delta u = f \quad \text{in} \quad B$$

$$\alpha u + \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial B, \alpha > 0$$

where n is unit outward normal to B. If a solution exist then,

(a) it is unique

(b) there are exactly two solutions

(c) there are exactly three solutions

(d) there are infinitely many solutions.

19. Consider the PDE

$$xu_x + yu_y = -xy \quad \text{for} \quad x > 0$$

subject to $u = 5$ on $xy = 1$ then

(a) $u(x,y)$ exists when $xy \le 19$ and $u(x,y) = u(y,x)$ for $x > 0, y > 0$

(b) $u(x, y)$ exists when $xy \geq 19$ and $u(x, y) = u(y, x)$ for $x > 0, y > 0$

(c) $u(1, 11) = 3, u(13, -1) = 7$

(d) $u(1, -1) = 5, u(11, 1) = -5$

20. If a complete integral of the PDE

$$x(p^2 + q^2) = zp, = z_x, q = z_y$$

passes through the curve $x = 0, z^2 = 4y$ then the envelope of this family passing through $x = 1$ and $y = 1$ has

(a) $z = -2$

(b) $z = 2$

(c) $z = \sqrt{2 + 2\sqrt{2}}$

(d) $z = -\sqrt{2 + 2\sqrt{2}}$

21. The PDE

$$xu_{xx} + yu_{yy} = 0$$

is

(a) Hyperbolic for $x > 0, y > 0$

(b) Elliptic for $x > 0, y < 0$

(c) Hyperbolic for $x > 0, y > 0$

(d) Elliptic for $x < 0, y > 0$

22. Let u be the solution of the following PDE

$$\begin{cases} u_t - u_{xx} = 0 & (x, t) \in (0, 1) \times (0, \infty) \\ u(x, 0) = \sin(\pi x) & x \in [0, 1] \\ u(0, t) = 0 & t > 0 \\ u(1, t) = 0 & t > 0 \end{cases}$$

Then the value of $u\left(x, \frac{1}{\pi^2}\right)$ is equal to

(a) $e \sin(\pi x)$

(b) $e^{-1} \sin(\pi x)$

(c) $\sin(\pi x)$

(d) $\sin(\pi^{-1} x)$

23. Let u be the unique solution of the following heat equation

$$\begin{cases} u_{tt} = u_{xx} & (x, t) \in \mathbb{R} \times (0, \infty) \\ u(x, 0) = f(x) & x \in \mathbb{R} \\ u_t(x, 0) = g(x) & x \in \mathbb{R} \end{cases}$$

Let u_i be the solution of the above problem with $f = f_i$ and $g = g_i$ for $i = 1, 2$ where $f_i, g_i \in C^2(\mathbb{R})$ satisfying $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$ in $x \in [-1, 1]$. Which of the following statements are necessarily true

(a) $u_1(0, 1) = u_2(0, 1)$

(b) $u_1(1, 1) = u_2(1, 1)$

(c) $u_1\left(\frac{1}{2}, \frac{1}{2}\right) = u_2\left(\frac{1}{2}, \frac{1}{2}\right)$

(d) $u_1(0, 2) = u_2(0, 2)$

24. The Cauchy problem

$$yu_x - xu_y = 0$$

and $u = g$ on Γ has a unique solution in the neighbourhood of Γ for every differentiable function $g : \Gamma \to \mathbb{R}$ if

(a) $\Gamma = \{(x, 0) : x > 0\}$

(b) $\Gamma = \{(x, y) : x^2 + y^2 = 1\}$

(c) $\Gamma = \{(x, y) : x + y = 1, x > 1\}$

(d) $\Gamma = \{(x, y) : y = x^2, x > 0\}$

25. Let a, b, c, d be four differentiable functions defined on $\mathbb{R}^2$. Then the PDE

$$(a(x, y)u_x + b(x, y)u_y)(c(x, y)u_x + d(x, y)u_y) = 0$$

is

(a) Always hyperbolic

(b) Always parabolic

(c) Never parabolic

(d) Never elliptic

26. Let u be the solution of the following heat equation

$$\begin{cases} u_t = u_{xx} & (x, t) \in \mathbb{R} \times (0, \infty) \\ u(x, 0) = x & x \in \mathbb{R} \end{cases}$$

Which of the following statements is/are true?

(a) The solution u exists for all $t > 0$

(b) The solution u exists for all $t < \frac{1}{2}$ and breaks down at $t = \frac{1}{2}$

(c) The solution u exists for all $t < 1$ and breaks down at $t = 1$

(d) The solution u exists for all $t < 2$ and breaks down at $t = 2$

27. Consider the Cauchy problem for the Eikonal equation

$$(p^2 + q^2) = 1, = u_x, q = u_y$$

$u(x, y) = 0$ on $x + y = 1, (x, y) \in \mathbb{R}^2$. Then

(a) The Charpit's equations for the differential equation are $x_t = 2p, y_t = 2q, u_t = 2, p_t = -p, q_t = -q$

(b) The Charpit's equations for the differential equation are $x_t = 2p, y_t = 2q, u_t = 2, p_t = 0, q_t = 0$

(c) $u(1, \sqrt{2}) = \sqrt{2}$

(d) $u(1, \sqrt{2}) = 1$

28. Let u be the solution of the following Laplace equation

$$\begin{cases} u_{xx} = u_{yy} & (x, y) \in (0, \pi) \times (0, \pi) \\ u(x, 0) = 0 & x \in (0, \pi) \\ u(x, \pi) = 0 & x \in (0, \pi) \\ u(0, y) = 0 & y \in (0, \pi) \\ u(\pi, y) = \sin y + \sin 2y & y \in (0, \pi) \end{cases}$$

Then which of the following is true?

(a) $u\left(1, \frac{\pi}{2}\right) = (\sinh(\pi))^{-1} \sinh(1)$

(b) $u\left(1, \frac{\pi}{2}\right) = (\sinh(1))^{-1} \sinh(\pi)$

(c) $u\left(1, \frac{\pi}{4}\right) = (\sinh(\pi))^{-1} \sinh(1) \frac{1}{\sqrt{2}} + (\sinh(2\pi))^{-1} \sinh(2)$

(d) $u\left(1, \frac{\pi}{4}\right) = (\sinh(1))^{-1} \sinh(\pi) \frac{1}{\sqrt{2}} + (\sinh(2))^{-1} \sinh(2\pi)$

29. For an arbitrary continuously differentiable function f, which of the following is a general solution of $z(px - qy) = y^2 - x^2$

(a) $x^2 + y^2 + z^2 = f(xy)$

(b) $(x + y)^2 + z^2 = f(xy)$

(c) $x^2 + y^2 + z^2 = f(y - x)$

(d) $x^2 + y^2 + z^2 = f((x + y)^2 + z^2)$

30. The PDE

$$u_{xx} + xu_{yy} = 0$$

is

(a) Elliptic for $x > 0$

(b) Hyperbolic for $x > 0$

(c) Elliptic for $x < 0$

(d) Hyperbolic for $x < 0$

31. Which of the following are complete integrals of the PDE $pqx + yq^2 = 1$?

(a) $z = \frac{x}{a} + \frac{ay}{x} + b$

(b) $z = \frac{x}{b} + \frac{ay}{x} + b$

(c) $z^2 = 4(ax + y) + b$

(d) $(z - b)^2 = 4(ax + y)$

32. Let $u(x,t) = e^{i\omega x}v(t)$ with $v(0) = 1$ be a solution to $u_t = u_{xxx}$, then

(a) $u(x,t) = e^{i\omega(x-\omega^2 t)}$

(b) $u(x,t) = e^{i\omega x - \omega^2 t}$

(c) $u(x,t) = e^{i\omega(x+\omega^2 t)}$

(d) $u(x,t) = e^{i\omega^3(x-t)}$

33. The Charpit's equation for the PDE

$$up^2 + q^2 + x + y = 0, p = u_x, q = u_y$$

are given by

(a) $\frac{dx}{-1-p^3} = \frac{dy}{-1-qp^2} = \frac{du}{2p^2u+2q^2} = \frac{dp}{2pu} = \frac{dq}{2q}$

(b) $\frac{dx}{2pu} = \frac{dy}{2q} = \frac{du}{2p^2u+2q^2} = \frac{dp}{-1-p^3} = \frac{dq}{-1-qp^2}$

(c) $\frac{dx}{up^2} = \frac{dy}{q^2} = \frac{du}{0} = \frac{dp}{x} = \frac{dq}{y}$

(d) $\frac{dx}{2q} = \frac{dy}{2pu} = \frac{du}{x+y} = \frac{dp}{p^2} = \frac{dq}{qp^2}$

34. Consider Cauchy problem for the Burger's equation

$$u_t + uu_x = 0, x \in \mathbb{R}, t > 0$$

and $u(x,0) = u_0(x), x \in \mathbb{R}$. Which choices of the following functions u_0 provides a C^1 solution for all $x \in \mathbb{R}, t > 0$

(a) $u_0(x) = \frac{1}{1+x^2}$

(b) $u_0(x) = x$

(c) $u_0(x) = 1 + x^2$

(d) $u_0(x) = 1 + 2x$

35. Let u satisfy the following PDE,

$$u_{tt} + u_t + 2u_{xx} = 0, x \in \mathbb{R}, t > 0$$

A solution of the form $u = e^{ix}v(t)$ with $v(0) = 0$ and $v'(0) = 1$

(a) is necessarily bounded

(b) satisfies $|u(x,t)| < e^t$

(c) is necessarily unbounded

(d) is oscillatory in x

36. The Cauchy Problem

$$xu_x + yu_y = 0$$

and $u(x,y) = x$ on $x^2 + y^2 = 1$ has

(a) a solution for all $x \in \mathbb{R}, y \in \mathbb{R}$

(b) an unique solution in $\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$

(c) a bounded solution in $\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$

(d) an unique solution in $\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$ but the solution is unbounded.

37. Let u be the solution of the following PDE

$$\begin{cases} u_t - u_{xx} = 0 & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = \sin(x) + \sin(2x) & x \in [0,\pi] \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) = 0 & t > 0 \end{cases}$$

Then

(a) $u(x,t) \to 0$ as $t \to \infty$ for all $x \in (0,\pi)$

(b) $t^2 u(x,t) \to 0$ as $t \to \infty$ for all $x \in (0,\pi)$

(c) $e^t u(x,t) \to 0$ is a bounded function for $x \in (0,\pi), t > 0$

(d) $e^{2t} u(x,t) \to 0$ as $t \to \infty$ for all $x \in (0,\pi)$

38. The solution of the Cauchy problem for the first order PDE

$$xz_x + yz_y = z \quad \text{on} \quad D = \{(x,y,z) : x^2 + y^2 \neq 0, z > 0\}$$

with the initial condition $x^2 + y^2 = 1, z = 1$ is

(a) $z = x^2 + y^2$

(b) $z = (x^2 + y^2)^2$

(c) $z = (2 - (x^2 + y^2))^{1/2}$

(d) $z = (x^2 + y^2)^{1/2}$

39. The PDE $u_{yy} - yu_{xx} = 0$ has

(a) Two families of real characteristic curves for $y > 0$

(b) no real characteristics for $y > 0$

(c) Vertical lines as a family of characteristic curves for $y = 0$

(d) Branches of quadratic curves as characteristics for $y \neq 0$

40. Let f be a function in $\mathbb{R}^2$ such that $f_x = f_y$ for all $(x, y) \in \mathbb{R}^2$. Then

(a) $f(x, y) - f(y, x) = (x - y)f_x(x^*, y^*) + (y - x)f_y(x^*, y^*)$ for some point $(x^*, y^*) \in \mathbb{R}^2$

(b) f is a constant on all lines parallel to the line $x = -y$

(c) $f(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$

(d) $f(x, y) = f(-y, x)$ for all $(x, y) \in \mathbb{R}^2$

41. Let the heat equation

$$u_t = \Delta u, t \geq 0, \mathbf{x} \in \mathbb{R}^3$$

admit an exponential function $e^{i\mathbf{k}.\mathbf{x}+\omega t}$ as its solution, where $\mathbf{k}$ is a nonzero constant real vector and ω is a constant. Then the solution

(a) Remains constant on certain planes in $\mathbb{R}^3$

(b) Repeats itself after a certain length L

(c) has in general an amplitude decaying exponentially with time t

(d) is bounded uniformly for $\mathbf{x} \in \mathbb{R}^3$ for a fixed t

42. Consider the Laplace equation in polar form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, 0 \leq r \leq a, 0 \leq \theta < 2\pi$$

satisfying $u(a, \theta) = f(\theta)$ where f is a given function. Let σ be the separation constant that appears when one uses the method of separation of variables. Then for solution $u(r, \theta)$ to be bounded and also periodic in θ with period 2π. Then

(a) σ cannot be negative

(b) σ can be zero and in that case the solution is a constant

(c) σ can be positive and in that case it must be an integer

(d) the fundamental set of solutions is $\{1, r^n \sin(n\theta), r^n \cos(n\theta)\}$

43. Consider the first order PDE

$$p + q = pq, p = z_z, q = z_y$$

Then which of the following are correct?

(a) The Charpit's equation for the above PDE reduce to

$$\frac{dx}{1-q} = \frac{dy}{1-p} = \frac{dz}{-pq} = \frac{dp}{p+q} = \frac{dq}{0}$$

(b) A solution of the Charpit's equation is $q = b$ where b is constant

(c) The corresponding value of p is $p = \frac{b}{b-1}$

(d) A solution of the equation is $z = \frac{b}{b-1}x + by + a$ where a and b are constants.

44. The PDE $yu_{xx} + xu_{yy} = 0$ is hyperbolic in

(a) the second and fourth quadrants

(b) the first and second quadrants

(c) the second and third quadrants

(d) the first and third quadrants

45. A bounded solution of the PDE

$$u_t = u_{xx} + e^{-t}$$

is

(a) $u(x,t) = -e^{-t}$

(b) $u(x,t) = e^{-x}e^{-t}$

(c) $u(x,t) = e^{-x} + e^{-t}$

(d) $u(x,t) = x - e^{-t}$

46. If $u(x,t)$ satisfy the PDE $u_{tt} = 4u_{xx}$ the $u(x,t)$ can be of the form

(a) $u(x,t) = f(e^{x-2t}) + g(x+2t)$

(b) $u(x,t) = f(x^2 - 4t^2) + g(x^2 + 4t^2)$

(c) $u(x,t) = f(2x - 4t) + g(x + 2t)$

(d) $u(x,t) = f(2x - t) + g(2x + t)$

47. The general solution of the PDE $uu_x + yu_y = x$ is of the form

(a) $f\left(u^2 - x^2, \frac{y}{x+u}\right) = 0$, where $f : \mathbb{R}^2 \to \mathbb{R}$ and $\nabla f \neq (0,0)$ at every point

(b) $u^2 = g\left(\frac{y}{x+u}\right) + x^2, g \in C^1(\mathbb{R})$

(c) $f(u^2 + x^2) = 0, f \in C^1(\mathbb{R})$

(d) $f(x + y) = 0, f \in C^1(\mathbb{R})$

48. The PDE

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0, & x, y \in (0,1) \\ u(x,0) = u(x,1) = 0 & x \in [0,1] \\ u(0,y) = u(1,y) = 0 & y \in [0,1] \end{cases}$$

has

(a) A unique solution for any $\lambda \in \mathbb{R}$

(b) infinitely many solutions for some $\lambda \in \mathbb{R}$

(c) A solution for countably many values of λ

(d) infinitely many solutions for all $\lambda \in \mathbb{R}$

49. The PDE

$$\begin{cases} u_x + u_y = 0, & x, y \in \mathbb{R} \\ u(x,x) = x^2 & x \in \mathbb{R} \end{cases}$$

has

(a) A unique solution

(b) A family of straight lines as characteristics

(c) Solution which vanishes at (2,1)

(d) infinitely many solutions

50. The general integral of the PDE

$$(2xy - 1)u_x + (z - 2x^2)u_y = 2(x - yz)$$

be given by $F(a, b) = 0$ where $F : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Then which of the following is true

(a) $a = x^2 + y^2 + u, b = xu + y$

(b) $a = x^2 + y^2 - z, b = xu - y$

(c) $a = x^2 - y^2 + u, b = yu + x$

(d) $a = x^2 + y^2 - z, b = yu - x$

51. If the characteristic curves of the PDE $xu_{xx} + 2x^2u_{xy} = u_x - 1$ are $\mu(x, y) = c_1$ and $\nu(x, y) = c_2$ where c_1 and c_2 are constants, then

(a) $\mu(x, y) = x^2 - y, \nu(x, y) = y$

(b) $\mu(x, y) = x^2 + y, \nu(x, y) = y$

(c) $\mu(x, y) = x^2 + y, \nu(x, y) = x^2$

(d) $\mu(x, y) = x^2 - y, \nu(x, y) = x^2$

52. If $u(x, y) = 1 + x + y + f(xy)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable function, then u satisfies

(a) $xu_x - yu_y = x^2 - y^2$

(b) $xu_x - yu_y = 0$

(c) $xu_x - yu_y = x - y$

(d) $yu_x - xu_y = x - y$

53. The PDE $xu_{xx} + (x - y)u_{xy} - yu_{yy} + \frac{1}{4}(u_y - u_x) = 0$ is ____

(a) Hyperbolic along the line $x + y = 0$

(b) Elliptic along the line $x + y = 0$

(c) Parabolic along the line $x + y = 0$

(d) Elliptic along the line $x - y = 0$

54. Let u be the solution of $xu_x + yu_y = 4u$ satisfying the conditions $u(x, y) = 1$ on the circle $x^2 + y^2 = 1$. Then $u(2, 2) =$ _____

55. Let $u(r, \theta)$ be the bounded solution of the following BVP in polar coordinates

$$ru_{rr} + ru_r + u_{\theta\theta} = 0, 0 < r < 2, 0 \leq \theta \leq 2\pi$$

and $u(2, \theta) = \cos^2\theta, 0 \leq \theta \leq 2\pi$. Then $u\left(1 + \frac{\pi}{2}\right) + u\left(1 + \frac{\pi}{4}\right)$ is _____

(a) 1

(b) $\frac{9}{8}$

(c) $\frac{7}{8}$

(d) $\frac{3}{8}$

56. Let u be the d'Alembert's solution of the IVP for the wave equation

$$u_{tt} - c^2 u_{xx} = 0, u(x, 0) = f(x), u_t(x, 0) = g(x)$$

where c is positive real constant. Suppose f and g are smooth odd functions. Then $u(0, 1) =$ _____

57. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$ with boundary $\partial\Omega$. If $u(x, y)$ is the solution of the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \text{ in } \Omega$$

and $u(x, y) = 1 - 2y^2$ on $\partial\Omega$. Then $u\left(\frac{1}{2}, 0\right)$ is _____

(a) -1

(b) $\frac{-1}{4}$

(c) $\frac{1}{4}$

(d) 1

58. Let $u(x, y) = 2f(y)\cos(x - 2y)$ be a solution of the IVP

$$2u_x + u_y = u$$

and $u(x, 0) = \cos(x)$ then $f(1)$ is _____

(a) $\frac{1}{2}$

(b) $\frac{e}{2}$

(c) e

(d) $\frac{3e}{2}$

59. Let u be the solution of the IVP $u_{tt} = u_{xx}, u(x, 0) = x, u_t(x, 0) = 1$, then $u(2, 2) =$

60. The integral surface of the first order PDE

$$2y(z - 3)z_x + (2x - z)z_y = y(2x - 3)$$

passing through the curve $x^2 + y^2 = 2x, z = 0$ is

(a) $x^2 + y^2 - z^2 - 2x + 4z = 0$

(b) $x^2 + y^2 - z^2 - 2x + 8z = 0$

(c) $x^2 + y^2 + z^2 - 2x + 16z = 0$

(d) $x^2 + y^2 + z^2 - 2x + 8z = 0$

61. Consider the heat equation

$$u_t - u_{xx} = 0, 0 < x < \pi, t > 0$$

with the boundary conditions $u(0, t) = 0, u(\pi, t) = 0$ for $t > 0$ and the initial condition $u(x, 0) = \sin x$. Then $u\left(\frac{\pi}{2}, 1\right) =$ _____

62. Let u be the solution of the wave equation

$$u_{tt} = u_{xx}, u(x, 0) = \cos(5\pi x), u_t(x, 0) = 0$$

Then the value of $u(1, 1) =$ _____

63. The integral surface of the PDE

$$z_x + z^2 z_y = 0$$

and passing through the straight line $x = 1, y = z$ is

(a) $(x - 1)z + z^2 = y^2$

(b) $x^2 + y^2 - z^2 = 1$

(c) $(y - z)x + x^2 = 1$

(d) $(x - 1)z^2 + z = y$

64. Consider the diffusion equation

$$u_t - u_{xx} = 0, 0 < x < \pi, t > 0$$

with the boundary conditions $u(0, t) = 0, u(\pi, t) = 0$ for $t > 0$ and the initial condition $u(x, 0) = \cos x \sin 5x$. Then $u(x, y) =$ _____

(a) $\dfrac{e^{-36t}}{2}\left[\sin 6x + e^{20t}\sin 4x\right]$

(b) $\dfrac{e^{-36t}}{2}\left[\sin 4x + e^{20t}\sin 6x\right]$

(c) $\dfrac{e^{-20t}}{2}\left[\sin 3x + e^{15t}\sin 5x\right]$

(d) $\dfrac{e^{-36t}}{2}\left[\sin 5x + e^{20t}\sin x\right]$

65. The PDE

$$x^2 z_{xx} - (y^2-1)xz_{xy} + y(y-1)^2 z_{yy} + xz_x + yz_y = 0$$

is hyperbolic in a region in the XY-plane if

(a) $x \neq$ and $y = 1$

(b) $x = 0$ and $y \neq 1$

(c) $x \neq 0$ and $y \neq 1$

(d) $x = 0$ and $y = 1$

66. The integral surface for the Cauchy problem

$$z_x + z_y = 1$$

which passes through the circle $z = 0, x^2 + y^2 = 1$ is

(a) $x^2 + y^2 + 2z^2 + 2zx - 2yz - 1 = 0$

(b) $x^2 + y^2 + 2z^2 + 2zx + 2yz - 1 = 0$

(c) $x^2 + y^2 + 2z^2 - 2zx - 2yz - 1 = 0$

(d) $x^2 + y^2 + 2z^2 + 2zx + 2yz + 1 = 0$

67. The vertical displacement $u(x,t)$ of an infinitely long elastic string is governed by the initial value problem

$$u_{tt} = 4u_{xx}, (x,t) \in \mathbb{R} \times (0,\infty)$$

$$u(x,0) = -x, u_t(x,0) = 0$$

The value of $u(x,t)$ at $x = 2$ and $t = 2$ is equal to

(a) 2

(b) 4

(c) -2

(d) -4

68. Consider the wave equation

$$u_{tt} = 4u_{xx}, (x,t) \in (0,\pi) \times (0,\infty)$$

with

$$u(0,t) = u(\pi,t) = 0, u(x,0) = \sin x, u_t(x,0) = 0$$

Then $u\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

(a) 2

(b) 1

(c) 0

(d) -1

69. The integral surface satisfying the equation $yz_x + xz_y = x^2 + y^2$ and passing through the curve $x = 1 - t, y = 1 + t, z = 1 + t^2$ is

(a) $z = xy + \frac{1}{2}(x^2 - y^2)^2$

(b) $z = xy + \frac{1}{4}(x^2 - y^2)^2$

(c) $z = xy + \frac{1}{8}(x^2 - y^2)^2$

(d) $z = xy + \frac{1}{16}(x^2 - y^2)^2$

70. For the diffusion problem

$$u_{xx} = u_t(x,t) \in (0,\pi) \times (0,\infty)$$

with

$$u(0,t) = u(\pi,t) = 0, u(x,0) = 3\sin 2x$$

the solution is given by

(a) $3e^{-t}\sin 2x$

(b) $3e^{-4t}\sin 2x$

(c) $3e^{-9t}\sin 2x$

(d) $3e^{-2t}\sin 2x$

71. Let $u(x,t)$ be the solution of $u_{tt} - u_{xx} = 1, x \in \mathbb{R}, t > 0$ with

$$u(x,0) = 0, u_t(x,0) = 0$$

Then $u\left(\frac{1}{2}, \frac{1}{2}\right) =$ _____

(a) $\frac{1}{8}$

(b) $-\frac{1}{8}$

(c) $\frac{1}{4}$

(d) $-\frac{1}{4}$

72. The initial value problem $u_x + u_y = 1, u(s,s) = \sin s, 0 \leq s \leq 1$ has

(a) two solutions

(b) a unique solution

(c) no solution

(d) infinitely many solutions

73. The characteristic curve $2yu_x + (2x + y^2)u_y = 0$ passing through $(0, 0)$ is

(a) $y^2 = 2(e^x + x - 1)$

(b) $y^2 = 2(e^x - x + 1)$

(c) $y^2 = 2(e^x - x - 1)$

(d) $y^2 = 2(e^x + x + 1)$

74. Consider the Neumann problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < \pi, -1 < y < 1 \\ u_x(0, y) = u_x(\pi, y) = 0 & -1 < y < 1 \\ u_y(x, -1) = 0, u_y(x, 1) = \alpha + \beta \sin x, 0 < x < \pi & \end{cases}$$

The problem admits solution for

(a) $\alpha = 0, \beta = 1$

(b) $\alpha = -1, \beta = \frac{\pi}{2}$

(c) $\alpha = 1, \beta = \frac{\pi}{2}$

(d) $\alpha = 1, \beta = -\pi$

75. Let $u(x, t)$ be the solution of the one dimensional wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & -\infty < x < \infty, t > 0 \\ u(x, 0) = \begin{cases} 16 - x^2 & |x| \le 4 \\ 0 & \text{otherwise} \end{cases} \quad u_t(x, 0) = \begin{cases} 1 & |x| \le 2 \\ 0 & \text{otherwise} \end{cases} & \end{cases}$$

For $1 < t < 3 u(2, t) =$

(a) $\frac{1}{2}[16 - (2 - 2t)^2] + \frac{1}{2}[1 - \min\{1, t - 1\}]$

(b) $\frac{1}{2}[32 - (2 - 2t)^2 - (2 + 2t)^2] + t$

(c) $\frac{1}{2}[32 - (2 - 2t)^2 - (2 + 2t)^2] + 1$

(d) $\frac{1}{2}[16 - (2 - 2t)^2] + \frac{1}{2}[1 - \max\{1 - t, -1\}]$

76. Let $u(x, t)$ be the solution of the one dimensional wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & -\infty < x < \infty, t > 0 \\ u(x, 0) = \begin{cases} 16 - x^2 & |x| \le 4 \\ 0 & \text{otherwise} \end{cases} \quad u_t(x, 0) = \begin{cases} 1 & |x| \le 2 \\ 0 & \text{otherwise} \end{cases} & \end{cases}$$

The value of $u_t(2, 2)$ is

(a) -15
(b) -16
(c) 0
(d) None

77. According to the classification of the second order linear differential operators, the following PDE

$$u_{xx} - 4u_{xy} + 5u_{yy} = 0$$

is of ____ type

78. A necessary and sufficient condition that the boundary value problem

$$u_{xx} + u_{yy} = f(x, y) \text{ in } \Omega$$

with

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

(where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$ and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of the function u has a solution is ____

79. Let

$$u_{xx} + u_{yy} = c \text{ in } \Omega$$

with

$$\frac{\partial u}{\partial \nu} = 1 \text{ on } \partial\Omega$$

(where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$ and $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of the function u. For what values of c does the above problem admit a solution?

80. Consider the Tricomi equation

$$u_{yy} - yu_{xx} = 0$$

. Describe the region in the $xy-$plane where this equation is elliptic.

81. Find d'Alembert's solution to the problem

$$\begin{cases} u_{tt} = u_{xx} & -\infty < x < \infty, t > 0 \\ u(x, 0) = x^2 & -\infty < x < \infty \\ u_t(x, 0) = 0 & -\infty < x < \infty \end{cases}$$

82. What is the necessary and sufficient condition for the following problem to admit a solution

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$, Δ is the Laplace operator, f and g are given smooth functions and $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of the function u.

83. Classify the following differential operator as elliptic, hyperbolic or parabolic

$$\mathcal{L}(u) = 2u_{xx} - 2u_{xy} + 2u_{yy}$$

84. Let B be the unit ball in the plane and let u be a solution of the boundary value problem

$$\begin{cases} -\Delta u = C & \text{in} \quad B \\ \frac{\partial u}{\partial \nu} = 1 & \text{on} \quad \partial B \end{cases}$$

Evaluate C, give that it is a constant.

85. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Write down the solution of the cccccccccccccccc

86. Classify the following PDE operators as elliptic, parabolic or hyperbolic

(a) $5u_{xx} + 6u_{xy} + 2u_{yy}$

(b) $2u_{xx} + 6u_{xy} + 2u_{yy}$

87. Solve

$$u_{xx} = 6xy, u(0, y) = y, u_x(1, y) = 0$$

88. Let $b \in \mathbb{R}$ be a constant. Solve

$$u_t + bu_x = 0x \in \mathbb{R}, t > 0$$

$$u(x, 0) = x^2$$

89. Let v be a smooth harmonic function on $\mathbb{R}^n$. If $r^2 = \sum_{i=1}^{n} |x_i|^2$ where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and if v is a radial function. That is, $v(x) = v(r)$ write down the ODE satisfied by v

90. Solve the BVP:

$$\begin{cases} -\Delta u = 1 & \text{in} \quad B(0, R) \\ u = 0 & \text{on} \quad \partial B(0, R) \end{cases}$$

where $B(0, R)$ is the open ball of radius R with centre at the origin.

91. The PDE $5u_{xx} + 6u_{yy} = xy$ is classified as

(a) elliptic

(b) parabolic

(c) hyperbolic

92. The PDE $xyu_x = 5u_{yy}$ is classified as

(a) elliptic

(b) parabolic

(c) hyperbolic

93. Consider the following PDE

$$\begin{cases} u_t = u_{xx} & 0 \le x \le 1, t > 0 \\ u(0,t) = 1 & t > 0 \\ u(1,t) = e^{-t} & t > 0 \\ u(x,0) = 1 & 0 \le x \le 1 \end{cases}$$

Find a function $\phi(x,t)$ such that $v(x,t) = u(x,t) + \phi(x,t)$ will satisfy the BCs

94. The function v from (93) will satisfy the PDE

$$\begin{cases} v_t = v_{xx} - xe^{-t} & 0 \le x \le 1, t > 0 \\ v(0,t) = 0 & t > 0 \\ v(1,t) = 0 & t > 0 \\ v(x,0) = 0 & 0 \le x \le 1 \end{cases}$$

95. Characteristics of the equation $y^2zp + zxq = y^2, p = z_x, q = z_y$ are

(a) $\frac{dx}{y^2z} = \frac{dy}{zx} = \frac{dz}{y^2}$

(b) $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{zx}$

(c) $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{zx}$

(d) $\frac{dx}{zx} = \frac{dy}{y^2z} = \frac{dz}{y^2}$

96. The PDE $u_{xx} + x^2u_{yy} = 0$ is classified as

(a) elliptic

(b) parabolic

(c) hyperbolic

97. The solution of the following IVP is

$$\begin{cases} u_{tt} = c^2u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = \sin x & x \in \mathbb{R} \\ u_t(x,0) = \cos x & x \in \mathbb{R} \end{cases}$$

(a) $\sin x \cos ct - \frac{1}{c} \cos x \sin ct$

(b) $\sin x \cos ct + \cos x \sin ct$

(c) $\sin x \cos ct + \frac{1}{c} \cos x \sin ct$

(d) $\sin x \cos ct$

98. Find the solution of the Cauchy problem

$$3u_x + 2u_y = 0, u(x,0) = \sin x$$

99. Find the general solution of the equation

$$u_x + 2xy^2 u_y = 0$$

100. Reduce the following equation to canonical form

$$u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$$