MA5023-Differential Equations for Engineers
Cheat Sheet

MA5023-Differential Equations for Engineers
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1 First Order ODE

First Order ODE:
Equation that contains only first derivative y' with or without y.

Implicit Form:

$$F(x,y,y') = 0 \tag{1}$$

Explicit Form:

$$y' = f(x,y) \tag{2}$$

Solution:
A function $y = h(x)$ is called a solution of given ODE (1) on some interval $a < x < b$ if $h(x)$ is defined and differentiable in that interval and is such that the equation becomes an identity if y and y' are replaced by h and h' respectively. The curve corresponds to h is called a solution curve.

General Solution:
A solution which contains an arbitrary constant is called a general solution of the ODE

Particular Solution:
If we choose a specific constant in the general solution, it is called a particular solution of the ODE.

Initial Value Problem (IVP):
An ODE in the explicit form (2), with initial condition $y(x_0) = y_0$

$$y' = f(x,y) \quad y(x_0) = y_0 \tag{3}$$

Separable ODE:

$$g(y)dy = f(x)dx$$

Exact ODE:

$$M(x,y)dx + N(x,y)dy = 0 \tag{4}$$

is called an exact ODE if the differential form $M(x,y)dx + N(x,y)dy$ is exact, that is, there exists some u such that

$$\frac{\partial u}{\partial x} = M \text{ and } \frac{\partial u}{\partial y} = N \tag{5}$$

Theorem on Exact ODE:
Suppose $M,N \in C^1(D), D = (a,b)\times(c,d)$. Then there exists ϕ such that

$$M = \frac{\partial \phi}{\partial x}, N = \frac{\partial \phi}{\partial y}$$

if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating Factor (I.F.):
If an ODE is of the form (4),then μ is said to be the integrating factor if $\mu M(x,y)dx + \mu N(x,y)dy$ is exact.

Theorem on I.F.
Suppose $Mdx + Ndy = 0$ and if μ is an integrating factor such that

$$\frac{1}{\mu}\frac{d\mu}{dx} = R(x)R(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\implies \mu = e^{\int R(x)dx}$$

First Order Linear ODE(FLODE):
A first-order ODE is said to be linear if it can be brought to the form

$$y' + p(x)y = r(x) \tag{6}$$

Otherwise, it is called first order nonlinear ODE.

Homogeneous, nonhomogeneous:
If $r(x) \equiv 0$ in (6), it is called homogeneous. Otherwise, it is called nonhomogeneous.

Solution for FLODE:

$$y(x) = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + c\right) \tag{7}$$

Solution for FLODE IVP:

$$y(x) = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)dx + y_0^*\right) \tag{8}$$

Bernoulli Equation:

$$y' + p(x)y = r(x)y^a$$

$$u = y^{1-a} \implies u' + (1-a)pu = (1-a)r$$

Existence Theorem:
Consider (3). Let $f(x,y)$ be continuous at all points (x,y) in some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded, that is there is a number K such that $|f(x,y)| \le K$ for all $(x,y) \in R$. Then the initial value problem has at least one solution $y(x)$. This solution exists at least for all x in the sub-interval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = min\{a, b/K\}$.

Uniqueness Theorem:
Consider (3). Let $f(x,y)$ and f_y be continuous at all points (x,y) in some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded, that is there is a number K such that $|f(x,y)| \le K$ and $|f_y(x,y)| \le M$ for all $(x,y) \in R$. Then the initial value problem has at most one solution $y(x)$. This solution exists at least for all x in the sub-interval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = min\{a, b/K\}$.

Remark: The condition $|f_y(x,y)| \le M$ can be replaced by a weaker condition or Lipschitz condition: $|f(x,y_1) - f(x,y_2)| \le M|y_2 - y_1|$ for all $(x,y_1),(x,y_2) \in R$.

2 Second Order ODE

Second-order Linear ODE (SLODE):
A second-order ODE is called linear if it can be written in the form

$$y'' + p(x)y' + q(x)y = r(x) \tag{9}$$

Homogeneous and nonhomogeneous:
If $r(x) \equiv 0$ in (9), then it is called second-order homogeneous linear ODE (SHLODE) (10), otherwise, it is called nonhomogeneous ODE (SNHLODE).

$$y'' + p(x)y' + q(x)y = 0 \tag{10}$$

Superposition Principle or Linearity Principle:
If y_1 and y_2 are any two solutions of the SHLODE (10) on an interval $[a,b]$, then any linear combination of y_1 and y_2, say $\alpha y_1 + \beta y_2$, for any $\alpha, \beta \in \mathbb{R}$, is also a solution of (10) in $[a,b]$

Initial Value Problem (IVP):

$$y'' + p(x)y' + q(x)y = 0 \tag{11}$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

Boundary Value Problem (BVP):

$$y'' + p(x)y' + q(x)y = 0, x \in [x_0, x_1], \tag{12}$$

$$y(x_0) = y_0, \quad y(x_1) = y_1$$

General and Particular Solution:
A solution $c_1y_1 + c_2y_2$ which contains arbitrary constants c_1 and c_2 is called a general solution. If we choose specific values for the constants, it is called particular solution.

Basis:
If the solutions of (10) are not proportional to each other, that is, if y_1 and y_2 are independent, then y_1, y_2 are called basis or fundamental system of (10).

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Proposition:
Let y_1, y_2 be any two solutions of (10), then $\alpha y_1 + \beta y_2$ is also a solution of (10), for any $\alpha, \beta \in \mathbb{R}$. Suppose y_1, y_2 are independent solution of (10), then any solution can be written in the form $y = \alpha y_1 + \beta y_2$,for some $\alpha, \beta \in \mathbb{R}$. If S denotes the set of all solutions of (10), then S is a linear space and $dim\ S \le 2$.

Wronskian:
Wronskian of two solutions y_1 and y_2 are defined as follows:

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1 y_2' - y_2 y_1' \quad (13)$$

Theorem:

1. If $p(x)$ and $q(x)$ of (10) are continuous in an open interval $[a,b]$, then two solutions y_1 and y_2 of (10) are linearly independent if and only if $W(y_1, y_2) \neq 0$ at some $x_0 \in [a,b]$.
2. $W \equiv 0$ or W is never zero.
3. $W \equiv 0$ if and only if y_1 and y_2 are dependent.
4. $dim\ S = 2$.

Existence and Uniqueness Theorem:
If $p(x)$ and $q(x)$ of (10) are continuous in an open interval $[a,b]$, then the IVP (11) has an unique solution $y(x)$ in the interval $[a,b]$.

Method to find Solution:
For the given SHLODE (10), assume the solution of this format $y = uv$. Step 1: Find u such that v' term vanishes, then solve for v using u from (14)

$$u = e^{-\frac{1}{2}\int p dx} \quad (14)$$

Reduction of Order Method:
Assume that one solution y_1 of (10) is known.Use $y_2(x) = c_1(x) y_1$. Compute y_2', y_2'' and solve $L y_2 = 0$. Assume $v = c'$ and solve (15) for v, then solve for c

$$\frac{c''}{c'} = -\frac{2y_1'}{y_1} - p \quad (15)$$

Constant Coefficients:
If $p(x), q(x)$ are constant in (10) and if r_1 and r_2 are the roots of the auxiliary or characteristic equation $\boldsymbol{r^2 + pr + q = 0}$, then general solution of (10) is given as follows:

$$y(x) = \begin{cases} c_1 e^{r_1 x} + c_2 e^{r_2 x} & r_1 \neq r_2 \in \mathbb{R} \\ (c_1 + c_2 x) e^{r_1 x} & r_1 = r_2 \in \mathbb{R} \\ e^{\frac{-\alpha}{2} x}(A cos \beta x + B sin \beta x) & r_1, r_2 \in \mathbb{C} \end{cases}$$

where $r_1 = \frac{\alpha}{2} + i\beta, r_2 = \frac{\alpha}{2} - i\beta$

Euler-Cauchy Equation:
An ODE of the form

$$x^2 y'' + axy' + by = 0 \quad (16)$$

is called Euler-Cauchy equation, where a and b are constants. Assume $y = x^m$, auxiliary equation, $m^2 + (a-1)m + b = 0$ and the general solution $y = c_1 x^{m_1} + c_2 x^{m_2}$

General and Particular solution of SNHLODE:
A general solution of SNHLODE (9) in an open interval $[a,b]$ is of the form $\boldsymbol{y(x) = y_h(x) + y_p(x)}$ where y_h is a general solution of (10) on $[a,b]$ and y_p is any solution of (10) without any arbitrary constants. If specific values are prescribed for c_1 and c_2, then the solution is called a particular solution.

Method of undetermined coefficients:

$$r(x) = \sum_{i=0}^{n} e^{\alpha x} x^i (k_i \cos \omega x + l_i \sin \omega x)$$

$$\implies y_p(x) = \sum_{i=0}^{n} e^{\alpha x} x^i (A_i \cos \omega x + B_i \sin \omega x)$$

Method of undetermined coefficients (Annihilator):

$$r(x) = x^n e^{\alpha x} \cos \omega x \ \text{ or } \ x^n e^{\alpha x} \cos \omega x$$

$$\implies A(D) = [(D-\alpha)^2 + \beta^2]^{n+1}$$

- Find the roots of $A(D)L(D) = 0$. Identify the roots of $L(D)$ and write the general solution
- Identify roots of $A(D)$ and write the particular solution y_p with constants.
- Solve $L(y_p) = r(x)$ to remove constants.

Method of Variation of Parameters:
Lagrange method gives a particular solution of (9) in the following form

$$y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx \quad (17)$$

Green's Function:

$$y_p(x) = \int_{x_0}^{x} \frac{-y_1(x) y_2(t) r(t)}{W} dt + \int_{x_0}^{x} \frac{y_2(x) y_1(t) r(t)}{W} dt$$

$$y_p(x) = \int_{x_0}^{x} G(x,t) r(t) dt$$

$$G(x,t) = \frac{y_2(x) y_1(t) - y_1(x) y_2(t) r(t)}{W} \quad (18)$$

(18) is called Green's function.

Green's Function for BVP:

$$G(x,t) = \begin{cases} \frac{y_2(x) y_1(t)}{W} & t \in [x_0, x] \\ \frac{y_1(x) y_2(t) r(t)}{W} & t \in [x, x_1] \end{cases}$$

and

$$y_p(x) = \int_{x_0}^{x_1} G(x,t) r(t) dt$$

3 Power Series Solution

Power Series:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \quad (19)$$

Convergent Power Series:
Partial sum of the power series is given by

$$S_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^N$$

(19) is convergent if $\lim_{N\to\infty} S_N(x)$ exists.

Interval of Convergence:
The set of all $x \in \mathbb{R}$ for which the power series converges. Converges definitely at x_0.

Radius of Convergence:
If the radius of the interval of convergence of a power series is called the radius of convergence.

- If $R > 0$, then the power series converges for $|x - x_0| < R$ and diverges for $|x - x_0| > R$.
- If the series converges only at x_0, then $R = 0$
- If the series converges for all x, then $R = \infty$
- Power series may or not converge at the endpoints of the interval

Absolute Convergence:
A series

$$\sum_{n=0}^{\infty} a_n$$

converges absolutely if the following converges

$$\sum_{n=0}^{\infty} |a_n|$$

A power series converges absolutely in its interval of convergence.

Ratio Test:
Suppose $a_n \neq 0$ for all n in the power series

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

- If $L < 1$, the series converges absolutely
- If $L > 1$, the series diverges
- If $L = 1$, no conclusion

Analytic:
A function f is said to be analytic at a point x_0 if it can be represented by a power series

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m \quad (20)$$

with either a positive or infinite radius of convergence. If the function is analytic at all points, we say f is analytic.

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Ordinary and Singular Points:
A point $x = x_0$ is said to be an **ordinary point** of the differential equation

$$y'' + p(x)y' + q(x)y = r(x) \quad (21)$$

if both coefficients $p(x)$ and $q(x)$ of (21) are **analytic** at x_0. A point that is *not an ordinary point* of the differential equation (21) is said to be a **singular point** of the ODE.

Regular and Irregular Singular Points:
A singular point $x = x_0$ is said to be a regular singular point of the differential equation (21) if the functions $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both (21) analytic at x_0. A point that is not regular is said to be an irregular singular point of the ODE.

Existence of Power Series Solution:
If p, q and r in (21) are analytic at $x = x_0$, then every solution of (21) is analytic at $x = x_0$ and can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Frobenius Method:
Let $b(x)$ and $c(x)$ be any functions that are analytic at $x = 0$. Then the ODE

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0 \quad (22)$$

has at least one solution that can be written in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad (23)$$

where the exponent r may be any real or complex number and r is chosen so that $a_0 \neq 0$. The ODE (22) has a second solution (such that these two solutions are linearly independent) that may be similar to (23) (with a different r and different coefficients) or may contain a logarithmic term.

Indicial Equation:
The indicial equation (22) is given by

$$r(r-1) + b_0 r + c_0 = 0 \quad (24)$$

Indicial Equation:
Suppose $b(x)$ and $c(x)$ of (22) are analytic at $x = 0$. Let r_1 and r_2 be the roots of the indicial equation (24). Then we have the following three cases
Case 1: Distinct roots not by differing by an integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \quad (25)$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \quad (26)$$

with coefficients obtained successively with $r = r_1$ and $r = r_2$ respectively.
Case 2: Double root $r_1 = r_2 = r$. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots), r = \frac{1}{2}(1 - b_0) \quad (27)$$

and

$$y_2(x) = y_1(x)\ln x + x^r(A_0 + A_1 x + A_2 x^2 + \cdots) \quad (28)$$

Case 3: Roots differing by an integer. A basis is

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \quad (29)$$

and

$$y_2(x) = k y_1(x)\ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \quad (30)$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

4 Legendre's Equation

Legendre's equation:

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \quad (31)$$

Any solution of Legendre equation is called Legendre function.

Indicial Equation:

$$(m+2)(m+1)a_{m+2} + [-m(m-1) - 2m + n(n+1)]a_m = 0$$

Recurrence Relation:

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)}a_m, m = 0, 1, \cdots$$

Solution of Legendre Equation:

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots$$

Legendre Polynomial:

$$P_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \quad (32)$$

where $\lfloor n/2 \rfloor$ denotes the floor function.

Rodrigue's Formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

Bonnet's recurrence relation:

$$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$$

Properties:
1. $P_n(-x) = (-1)^n P_n(x)$
2. $P_n(-1) = (-1)^n$
3. $P_{2n+1}(0) = P'_{2n}(0) = P''_{2n+1}(0) = 0$
4. $P_{2n}(0) = (-1)^n \frac{1.3.5.\cdots.(2n-1)}{2.4.6.\cdots.2n}$
5. $P'_{2n+1}(0) = (-1)^n \frac{1.3.5.\cdots.(2n-1)(2n+1)}{2.4.6.\cdots.2n}$
6. $P'_n(1) = \frac{n(n+1)}{2}$
7. $P'_n(-1) = (-1)^{n-1}\frac{n(n+1)}{2}$
8. $P'_{n+1} + P'_{n-1} = 2xP'_n + P_n$
9. $P'_{n+1} = xP'_n + (n+1)P_n$
10. $P'_{n+1} - P'_{n-1} = (2n+1)P_n$
11. $P'_{n-1} = xP'_n - nP_n$
12. $(1 - x^2)P'_n = nP_{n-1} - nxP_n$
13. $(1 - x^2)P'_n = (n+1)xP_n - (n+1)P_{n+1}$

Orthogonality

$$\int_{-1}^{1} P_n P_m dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m \end{cases}$$

Legendre Functions of the second kind

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x}$$

Properties of $Q_n(x)$
1. $Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$
2. $(n+1)Q_{n+1} = (2n+1)xQ_n - nQ_{n-1}$
3. $Q_{2n}(0) = 0$
4. $Q_{2n}(1) = \infty$
5. $Q_n(x) = \sum_{m=1}^{n} P_{m-1}(x)P_{n-m}(x)$

5 Bessel's Equation

Bessel's equation:

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \quad (33)$$

Recurrence Relation for $r = \nu$**:**

$$a_m = -\frac{1}{m(m+2\nu)}a_{m-2}$$

$$a_{2k} = \frac{(-1)^k}{2^{2k}k!(\nu+k)(\nu+k-1)\cdots(\nu+2)(\nu+1)}a_0$$

Bessel Function of the first kind $J_n(x)$**:**

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k}{k!(n+k)!}$$

Bessel Function of the first kind $J_\nu(x)$**:**

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}$$

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Bessel Function of the first kind $J_{-\nu(x)}$:

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k} \frac{(-1)^k}{k!\Gamma(-\nu+k+1)}$$

Bessel Function of the Second kind $Y_\nu(x)$:

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)]$$

Properties of $J_\nu(x)$ and $Y_n(x)$:

1. $[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$
2. $[x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x)$
3. $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$
4. $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu^{'}(x)$
5. $xJ_\nu^{'}(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$
6. $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$
7. $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$
8. $J_{-n}(x) = (-1)^n J_n(x)$
9. $$J_n(0) = \begin{cases} 1 & n > 0 \\ 0 & n = 0 \end{cases}$$
10. $$\int_0^1 xJ_n(\lambda x)J_n(\mu x) = \begin{cases} 0 & \lambda \neq \mu 0 \\ \frac{1}{2}J_{n+1}^2 \lambda & \lambda = \mu \end{cases}$$
11. $\lim_{x\to 0} Y_n(x) = -\infty$
12. $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$
13. $\sin x = 2\sum_{n=1}^{\infty}(-1)^n J_{2n+1}(x)$

6 Fourier Series

Orthonormality of Trigonometric system:

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

Fourier Series:
Suppose $f(x)$ is a given function of period 2π

$$f(x) = a_0 + \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx) \quad (34)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

Fourier Series (period 2L):

$$f(x) = a_0 + \sum_{n=0}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x) \quad (35)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x dx$$

Representation by a Fourier Series:
Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $[-\pi, \pi]$. Furthermore, let $f(x)$ have a left-hand and right-hand derivative at each point of that interval. Then the Fourier series of $f(x)$ converges. Its sum is $f(x)$, except at points x_0 where $f(x)$ is continuous. There sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Fourier Series (f is odd and $L = \pi$):

$$f(x) = \sum_{n=0}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi}\int_{-L}^{L} f(x)\sin nx dx$$

Fourier Series (f is even and $L = \pi$):

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos nx \quad (36)$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

Sturm-Liouville Problem:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, x \in [a,b]$$
$$k_1 y(a) + k_2 y'(a) = 0$$
$$l_1 y(b) + l_2 y'(b) = 0$$

Here λ is a parameter and k_i, l_i are constants.

Orthogonality of Eigenfunction:
Suppose that the function p, q, r, p' in Sturm-Liouville problem are real valued and continuous and $r(x) > 0$ on $[a,b]$. Let y_m, y_n be eigentfunctions of Sturm-Liouville problem corresponding to eigenvalues λ_m, λ_n respectively. Then y_m, y_n are orthogonal on $[a,b]$ w.r.t. to the weight function r. That is

$$(y_m, y_n) = \int_a^b r(x)y_m(x)y_n(x)dx = 0, \quad m \neq n$$

Generalized Fourier Series:
Let $y_0, y_1, y_2, \cdots$ be orthogonal w.r.t. a weight function $r(x)$ on $[a,b]$. Let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n y_n(x) \quad (37)$$

It is also orthogonal expansion. If y_n's are eigenfunctions of the Sturm-Liouville problem, then it is called as an eigenfunction expansion.

Fourier Integral:

$$f(x) = \int_{n=0}^{\infty} (A(w)\cos wx + B(w)\sin wx)dw \quad (38)$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv dv$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv dv$$

Fourier Cosine Integral:

$$f(x) = \int_0^{\infty} A(w)\cos wx dw$$

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos wv dv$$

Fourier Sine Integral:

$$f(x) = \int_0^{\infty} B(w)\sin wx dw$$

$$B(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin wv dv$$

7 Partial Differential Equations

First order Linear/Nonlinear PDE in 2D:
The first-order linear PDE is of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + f(x,y)$$

First-order Semilinear PDE in 2D:
The first-order semilinear PDE is of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

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First-order Quasilinear PDE in 2D:
The first-order quasilinear PDE is of the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

Second-order Quasilinear PDE:
Consider the following PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} = \psi(x,y,u,u_x,u_y) \quad (39)$$

PDE Classification:
The second-order quasilinear PDE is classified into three types depending on the discriminant B^2-4AC

$$\begin{cases} B^2-4AC>0 & \text{Hyperbolic} \\ B^2-4AC<0 & \text{Elliptic} \\ B^2-4AC=0 & \text{Parabolic} \end{cases}$$

PDE Canonical Form:
Steps to obtain canonical form

1. Introduce new variables $\xi = \xi(x,y), \eta = \eta(x,y)$.
2. Transform (x,y) to (ξ,η)
3. Obtain $au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} = \phi(\xi,\eta,u_\xi,u_\eta)$
4. $a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$
5. $b = 2A\xi_x\eta_x + B()\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y$
6. $c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$

Type	Cases	Canonical
Hyperbolic	$a=c=0$	$u_{\xi\eta}=\phi$
Hyperbolic	$b=0, c=-a$	$u_{\alpha\alpha}-u_{\beta\beta}=\phi_1$
Parabolic	$a=b=0$	$u_{\eta\eta}=\phi$
Elliptic	$b=0, c=a$	$u_{\xi\xi}+u_{\eta\eta}=\phi$

Here $\alpha=\xi+\eta, \beta=\xi-\eta, \phi=\phi(\xi,\eta,u_\xi,u_\eta)$ and $\phi_1=\phi_1(\alpha,\beta,u_\alpha,u_\beta)$

Four Important PDEs:

$$u_t + b.Du = 0 \quad \text{Transport Equation}$$
$$u_t = c^2\Delta u \quad \text{Heat Equation}$$
$$u_{tt} = c^2\Delta u \quad \text{Wave Equation}$$
$$\Delta u = 0 \quad \text{Laplace Equation}$$

Wave Equation Solution (1D):

$$\begin{cases} u_{tt} = c^2u_{xx} & 0<x<L, t>0 \\ u(0,t)=u(L,t)=0 & t>0 \\ u(x,0)=f(x), u_t(x,0)=g(x), & 0\le x\le L \end{cases} \quad (40)$$

Wave Equation Solution (Variable Separable):
Steps to Solve (40)

1. Assume $u(x,t)=F(x)G(t)$
2. Find u_{tt}, u_{xx}
3. Find two ODEs with BVP that satisfy the Boundary conditions
4. Solve two ODEs
5. Using the Fourier series, compose the solution that satisfy both Boundary and Initial Conditions

Wave Equation Solution (d'Alembert's):

1. Introduce variable $v=x+ct, w=x-ct$
2. Transform wave equation from (x,t) to (v,w) and obtain $u_{vw}=0$
3. Solve and use initial conditions to obtain

$$u(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Heat Equation Solution (1D):

$$\begin{cases} u_t = c^2u_{xx} & 0<x<L, t>0 \\ u(0,t)=u(L,t)=0 & t>0 \\ u(x,0)=f(x), & 0\le x\le L \end{cases} \quad (41)$$

Heat Equation Solution (Variable Separable):
Steps to Solve (41)

1. Assume $u(x,t)=F(x)G(t)$
2. Find u_t, u_{xx}
3. Find two ODEs with BVP that satisfy the Boundary conditions
4. Solve two ODEs
5. Using the Fourier series, compose the solution that satisfy both Boundary and Initial Conditions

Laplace Equation Solution (2D):

$$\begin{cases} u_{xx}+u_{yy}=0 & 0<x<a, 0<y<b \\ u(0,y)=u(a,y)=0 & 0\le y\le b \\ u(x,0)=f(x), u(x,b)=0, & 0\le x\le a \end{cases} \quad (42)$$

Laplace Equation Solution (Variable Separable):
Steps to Solve (42)

1. Assume $u(x,y)=F(x)G(y)$
2. Find u_{xx}, u_{yy}
3. Find two ODEs with BVP that satisfy the Boundary conditions
4. Solve two ODEs
5. Using the Fourier series, compose the solution that satisfy Boundary Conditions

Wave Equation Solution (2D):

$$\begin{cases} u_{tt} = c^2(u_{xx}+u_{yy}) & (x,y)\in(a,b), t>0 \\ u(x,0,t)=u(x,b,t)=0 & t>0 \\ u(0,y,t)=u(a,b,t)=0 & t>0 \\ u(x,y,0)=f(x,y) & 0\le x\le a, 0\le y\le b \\ u_t(x,y,0)=g(x,y) & 0\le x\le a, 0\le y\le b \end{cases} \quad (43)$$

Wave Equation Solution (Variable Separable):
Steps to Solve (43)

1. Assume $u(x,y,t) = F(x,y)G(t) = H(x)Q(y)G(t)$
2. Find u_{tt}, u_{xx}, u_{yy}
3. Find $F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$
4. Solve $\ddot{G}+\lambda^2G=0, \lambda=cv$
5. Solve the Helmholtz equation $F_{xx}G+F_{yy}G+v^2F=0$ using $F(x,y)=H(x)Q(y)$
6. Find $H''+k^2H=0$
7. Find $Q''+p^2Q=0, p^2=v^2-k^2$
8. Using boundary conditions, obtain $F_{mn}(x,y)$
9. Using Double Fourier Series, obtain the solution that satisfy both Boundary and Initial Conditions.

Laplace Equation (Polar):

$$\begin{cases} r=\sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \\ u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \end{cases} \quad (44)$$

Laplace Equation (Cylindrical):

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \\ u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+u_{zz}=0 \end{cases} \quad (45)$$

Laplace Equation (Spherical):

$$\begin{cases} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{cases} \quad (46)$$

$$0 = u_{rr}+\frac{2}{r}u_r+\frac{1}{r^2}u_{\phi\phi}+\frac{\cot\phi}{r^2}u_\phi+\frac{1}{r^2\sin^2\phi}u_{\theta\theta} \quad (47)$$

Wave Equation (Polar):

$$\begin{cases} u_{tt}=c^2\left(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}\right) \\ u(R,t)=0, \\ u(r,0)=f(r), \quad u_t(r,0)=g(r) \end{cases} \quad (48)$$

Wave Equation Solution (Variable Separable):
Steps to Solve (48)

1. Assume $u(r,t)=W(r)G(t)$
2. Find u_{tt}, u_{rr}
3. Obtain $\frac{\ddot{G}}{c^2G} = \frac{1}{W}(W''+\frac{1}{r}W') = -k^2$
4. Solve $\ddot{G}+\lambda^2G=0, \lambda=ck$
5. Obtain Bessel equation $W''+\frac{1}{r}W'+k^2W=0$
6. Using boundary conditions, obtain $W(r)$
7. Using Fourier-Bessel Series, obtain the solution that satisfy both Boundary and Initial Conditions.