INDIAN INSTITUTE OF TECHNOLOGY TIRUPATI
DEPARTMENT OF MATHEMATICS AND STATISTICS

MA5023-DIFFERENTIAL EQUATIONS FOR ENGINEERS

Questions and Answers

1. Without solving the initial value problem, what is the largest interval in which a unique solution is guaranteed to exist for the following IVP?

$$y' = \frac{x-2}{x^2(x+3)} + \sec\left(\frac{x}{3}\right)$$

   (a) $y(-4) = e$
   (b) $y(1) = -9$
   (c) $y(\pi) = 7$

Solution: The IVP can be written as

$$y' = f(x, y)$$

where

$$f(x, y) = \frac{x-2}{x^2(x+3)} + \sec\left(\frac{x}{3}\right)$$

According to the following theorem

Uniqueness Theorem: Consider the following initial value problem

$$y' = f(x, y), y(x_0) = y_0 \tag{1}$$

Suppose, $f(x, y)$ is continuous for all points (x, y) in some rectangle

$$R : |x - x_0| < a, |y - y_0| < b \tag{2}$$

and bounded in R, that is there exists a number K such that

$$|f(x, y)| \leq K, \text{ for all } (x, y) \in R \tag{3}$$

Further, of partial derivative of f w.r.to y (that is $f_y = \frac{\partial f}{\partial y}$), is continuous for all $(x, y) \in R$ and bounded, say

$$|f_y(x, y)| \leq M, \text{ for all } (x, y) \in R \tag{4}$$

for some M. Then the initial value problem has at most one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = \min\{a, b/K\}$.

Step 1: We need to find whether $f(x, y)$ is continuous for all points in some rectangle first as in equation (2).

$f(x, y)$ is independent of y. We do not care about y at the moment. However, our rectangle should contain x_0, y_0. Therefore, the interval of y is $\mathbb{R}$. Now, f can be written as

$$f(x, y) = g(x) + h(x)$$

where

$$g(x) = \frac{x-2}{x^2(x+3)}$$

and

$$h(x) = \sec\left(\frac{x}{3}\right)$$

Now, g is not defined at $x = 0$ and $x = -3$. Therefore, g is continuous in the interval $(-\infty, -3)$, $(-3, 0)$ and $(0, \infty)$

Now, $\cos x$ is 0 whenever $x = n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$. Therefore, $\sec x$ is defined for all $x \in \mathbb{R}$ except for $x = n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$. Hence $h(x)$ is defined for all $x \in \mathbb{R}$ except for $\frac{x}{3} = n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$. That is, h is not defined for all $x = 3n\pi + \frac{3\pi}{2} = \frac{3(2n-1)\pi}{2}$. In other words, h is defined for all $(-\frac{3(2k+1)\pi}{2}, -\frac{3(2k-1)\pi}{2}), \cdots, (-\frac{9\pi}{2}, -\frac{3\pi}{2}), (-\frac{3\pi}{2}, \frac{3\pi}{2}), (\frac{3\pi}{2}, \frac{9\pi}{2}), \cdots, (\frac{3(2k-1)\pi}{2}, \frac{3(2k+1)\pi}{2})$. Length of each of these interval is 2π. Therefore, $g + h$ is continuous in the interval $(-\frac{3(2k+1)\pi}{2}, -\frac{3(2k-1)\pi}{2}), \cdots, (-\frac{9\pi}{2}, -\frac{3\pi}{2}),(-\frac{3\pi}{2}, -3),(-3, 0), (0, \frac{3\pi}{2}), (\frac{3\pi}{2}, \frac{9\pi}{2}), \cdots, (\frac{3(2k-1)\pi}{2}, \frac{3(2k+1)\pi}{2})$.

Step 2: Note that $f_y = 0$ and b is unbounded and hence $\alpha = a$. Therefore, our job is to find only largest sub-interval where x_0 lies.

For, problem (a), $x_0 = -4, y_0 = e$. Here, $x_0 \in (-\frac{3\pi}{2}, -3)$. Also, f is valid and bounded in the interval

$$\left(-\frac{3\pi}{2}, -3\right).$$

Therefore, the largest interval is

$$\left(-\frac{3\pi}{2}, -3\right)$$

For, problem (b), $x_0 = 1, y_0 = -9$.Here, $x_0 \in (0, \frac{3\pi}{2})$. Also, f is valid and bounded in the interval

$$\left(0, \frac{3\pi}{2}\right).$$

Therefore, the largest interval is

$$\left(0, \frac{3\pi}{2}\right)$$

For, problem (c), $x_0 = \pi, y_0 = 7$.Here, $x_0 \in (0, \frac{3\pi}{2})$. Also, f is valid and bounded in the interval

$$\left(0, \frac{3\pi}{2}\right).$$

Therefore, the largest interval is

$$\left(0, \frac{3\pi}{2}\right)$$

2. Use the method of reduction of order to find a second solution of the given differential equation:

$$x^2y'' + 2xy' - 2y = 0$$

$$y_1(x) = x$$

Solution: When y_1 is known, by reduction of order method, we can find the second solution by

$$y_2(x) = y_1(x)\int \frac{e^{-\int p dx}}{y_1^2(x)}dx$$

Here

$$y_1 = x, p = \frac{2}{x}$$

Therefore,

$$\int p dx = \int \frac{2}{x} = 2\ln x$$

Hence

$$\begin{aligned} y_2(x) &= y_1(x)\int \frac{e^{-\int p dx}}{y_1^2(x)}dx \\ &= x\int \frac{e^{-2\ln x}}{x^2}dx \\ &= x\int \frac{1}{x^4}dx \\ &= x\frac{1}{-3x^3} \\ &= \frac{1}{-3x^2} \end{aligned}$$

Hence, a second solution is given by

$$y_2 = \frac{1}{x^2}$$

3. Use the method of reduction of order to find the second solution of the given differential equation:

$$x^2y'' + 7xy' + 9y = 0$$

$$y_1(x) = x^{-3}$$

Solution: When y_1 is known, by reduction of order method, we can find the second solution by

$$y_2(x) = y_1(x) \int \frac{e^{-\int p dx}}{y_1^2(x)} dx$$

Here

$$y_1 = x^{-3}, p = \frac{7}{x}$$

Therefore,

$$\int p dx = \int \frac{7}{x} = 7 \ln x$$

Hence

$$\begin{aligned} y_2(x) &= y_1(x) \int \frac{e^{-\int p dx}}{y_1^2(x)} dx \\ &= x^{-3} \int \frac{e^{-7 \ln x}}{x^{-6}} dx \\ &= x^{-3} \int \frac{1}{x} dx \\ &= \frac{\ln x}{x^3} \end{aligned}$$

Hence, a second solution is given by

$$y_2 = \frac{\ln x}{x^3}$$

4. Use the method of reduction of order to find a second solution of the given differential equation:

$$x^2 y'' + 6xy' + 6y = 0$$

$$y_1(x) = x^{-2}$$

Solution: When y_1 is known, by reduction of order method, we can find the second solution by

$$y_2(x) = y_1(x) \int \frac{e^{-\int p dx}}{y_1^2(x)} dx$$

Here

$$y_1 = x^{-2}, p = \frac{6}{x}$$

Therefore,

$$\int p dx = \int \frac{6}{x} = 6 \ln x$$

Hence

$$\begin{aligned} y_2(x) &= y_1(x)\int \frac{e^{-\int p dx}}{y_1^2(x)}dx \\ &= x^{-2}\int \frac{e^{-6\ln x}}{x^{-4}}dx \\ &= x^{-2}\int \frac{1}{x^2}dx \\ &= x\frac{1}{-x} \\ &= \frac{1}{-x^3} \end{aligned}$$

Hence, a second solution is given by

$$y_2 = \frac{1}{x^3}$$

5. Consider the following second-order linear ODE:

$$x^2y'' + y'' - 4xy' + 6y = 0$$

(a) Find the ordinary point, singular point, regular singular point and irregular singular points for the following equation.

Solution: The above equation can be written as

$$(x^2+1)y'' - 4xy' + 6y = 0$$

or

$$y'' - \frac{4x}{x^2+1}y' + \frac{6}{x^2+1}y = 0$$

All points in the real line are ordinary points for this problem. In complex plane, $x = \pm i$ are singular points. Both are regular singular points as $(x \pm i)\frac{4x}{x^2+1}$ and $(x \pm i)^2\frac{4x}{x^2+1}$ are analytic at $x = \pm i$.

(b) Using the power series

$$y = \sum_{m=0}^{\infty} a_m x^m$$

find two independent solutions.

Solution:

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} a_m x^m \\ y'(x) &= \sum_{m=1}^{\infty} m a_m x^{m-1} \\ y''(x) &= \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} \end{aligned}$$

$$
\begin{aligned}
x^2y'' + y'' - 4xy' + 6y &= x^2\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} \\
&- 4x\sum_{m=1}^{\infty} m a_m x^{m-1} + 6\sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=2}^{\infty} m(m-1)a_m x^{m} + \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} \\
&- 4\sum_{m=1}^{\infty} m a_m x^{m} + 6\sum_{m=0}^{\infty} a_m x^m
\end{aligned}
$$

$$
\begin{aligned}
0 &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m} + \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^{m} \\
&- 4\sum_{m=1}^{\infty} m a_m x^{m} + 6\sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=2}^{\infty} [m(m-1)a_m + (m+2)(m+1)a_{m+2} - 4m a_m + 6a_m] x^m \\
&+ 2a_2 + 6a_3 x - 4a_1 x + 6a_0 + 6a_1 x \\
&= \sum_{m=2}^{\infty} [(m(m-1) - 4m + 6)a_m + (m+2)(m+1)a_{m+2}] x^m \\
&+ (2a_2 + 6a_0) + (6a_3 + 2a_1)x
\end{aligned}
$$

Comparing constant terms, we get

$$
2a_2 + 6a_0 = 0 \implies a_2 = -3a_0
$$

Comparing x coefficient

$$
6a_3 + 2a_1 = 0 \implies a_3 = -\frac{1}{3}a_1
$$

Comparing x^m coefficient

$$
(m^2 - 5m + 6)a_m + (m+2)(m+1)a_{m+2} = 0
$$

$$
\implies a_{m+2} = -\frac{(m-3)(m-2)}{(m+2)(m+1)}a_m
$$

When $m = 3, a_5 = 0.a_3 = 0$ When $m = 2, a_4 = 0.a_2 = 0$ Therefore, except first four coefficients, rest are all zero.

Therefore the solution is given by

$$
y = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3}a_1 x^3
$$

$$\implies y = a_0(1 - 3x^2) + a_1\left(x - \frac{1}{3}x^3\right)$$

Therefore, the two independent solutions are

$$y_1 = 1 - 3x^2$$

and

$$y_2 = x - \frac{1}{3}x^3$$

6. Bessel's Function and Legendre Polynomial.

   (a) Write down the Bessels Functions of the first kind $J\nu(x)$ and Legendre Polynomial $P_n(x)$ Bessels Functions of the first kind $J\nu(x)$

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k}{k!\Gamma(\nu+k+1)} \tag{5}$$

   Legendre Polynomial $P_n(x)$

$$P_n(x) = \sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m} \tag{6}$$

   (b) Prove one of the following:

      i. $[x^{-\nu}J_\nu(x)]' = -x^{-\nu}J_{\nu+1}(x)$

      ii. $[x^{\nu}J_\nu(x)]' = x^{\nu}J_{\nu-1}(x)$

      iii. $(1-x^2)P_n' = (n+1)xP_n - (n+1)P_{n+1}$

      iv. $P_{n+1}' - P_{n-1}' = (2n+1)P_n$

   Solution for (i): Multiply equation (5) by $x^{-\nu}$ and differentiate

$$x^{-\nu}J_\nu(x) = x^{-\nu}\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}$$

$$[x^{\nu}J_\nu(x)]' = \cancelto{1}{x^{-\nu}\, x^{\nu}}\left(\frac{1}{2}\right)^\nu \sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^{2k-1}\frac{(-1)^k(2k+\nu-\nu)}{k!\Gamma(\nu+k+1)}$$

$$[x^{\nu}J_\nu(x)]' = x^{-\nu}\left(\frac{x}{2}\right)^\nu \sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^{2k-1}\frac{(-1)^k k}{k!\Gamma(\nu+k+1)}$$

   Solution for (ii): Multiply equation (5) by x^ν and differentiate.

$$x^{\nu}J_\nu(x) = x^{\nu}\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}$$

$$[x^{\nu}J_\nu(x)]' = x^{\nu-1}\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k(2k+2\nu)}{k!\Gamma(\nu+k+1)}$$

$$[x^{\nu}J_\nu(x)]' = x^{\nu-1}\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k 2(k+\nu)}{k!(\nu+k)\Gamma(\nu+k)}$$

$$[x^{\nu}J_\nu(x)]' = x^{\nu}\left(\frac{x}{2}\right)^{\nu-1} \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{(-1)^k}{k!\Gamma(\nu+k)} = x^\nu J_{\nu-1}(x)$$

Solution for (iii) and (iv): For this, we need the following properties.

(a) $P'_{n+1} + P'_{n-1} = 2xP'_n + P_n$

(b) $P'_{n+1} = xP'_n + (n+1)P_n$

(c) $P'_{n-1} = xP'_n - nP_n$

Using (a) in (b), we obtain that

$$P'_{n+1} - \frac{P'_{n+1} + P'_{n-1} - P_n}{2} - (n+1)P_n = 0$$
$$\frac{P'_{n+1}}{2} - \frac{P'_{n-1}}{2} - (n + \frac{1}{2})P_n = 0$$
$$P'_{n+1} - P'_{n-1} = (2n+1)P_n \tag{7}$$

Hence (iv) follows.

In (c), replacing n by $n+1$, we get

$$P'_n = x\, P'_{n+1} - (n+1)P_{n+1} \tag{8}$$

Using (b) in (8), proves (iii).

$$P'_n = x(xP'_n + (n+1)P_n) - (n+1)P_{n+1}$$
$$(1-x^2)P'_n = (n+1)xP_n - (n+1)P_{n+1}$$