1. Laplace Equation in Spherical Coordinates.

$$x = r\cos\theta\sin\phi\,, y = r\sin\theta\sin\phi\,, z = r\cos\phi$$

Method 1:

$$x_r = \cos\theta\sin\phi = \frac{x}{r}, y_r = \sin\theta\sin\phi = \frac{y}{r}, z_r = \cos\phi = \frac{z}{r}$$
$$x_\theta = -r\sin\theta\sin\phi = -y, y_\theta = r\cos\theta\sin\phi = x, z_\theta = 0$$
$$x_\phi = r\cos\theta\cos\phi\,, y_\phi = r\sin\theta\cos\phi\,, z_\phi = -r\sin\phi$$
$$u_r = u_x x_r + u_y y_r + u_z z_r$$
$$ru_r = u_x r\cos\theta\sin\phi + u_y r\sin\theta\sin\phi + u_z r\cos\phi$$
$$u_\theta = u_x x_\theta + u_y y_\theta$$
$$u_\theta = -u_x r\sin\theta\sin\phi + u_y r\cos\theta\sin\phi$$
$$u_\phi = u_x x_\phi + u_y y_\phi + u_z z_\phi$$
$$u_\phi = u_x r\cos\theta\cos\phi + u_y r\sin\theta\cos\phi - u_z r\sin\phi$$
$$x_{rr} = 0, y_{rr} = 0, z_{rr} = 0$$
$$x_{\theta\theta} = -y_\theta = -x, y_{\theta\theta} = x_\theta = -y, z_{\theta\theta} = 0$$
$$x_{\phi\phi} = -r\cos\theta\sin\phi = -x, y_{\phi\phi} = -r\sin\theta\sin\phi = -y, z_{\phi\phi} = -r\cos\phi = -z$$
$$u_{rr} = \left(u_x x_r + u_y y_r + u_z z_r\right)_r$$
$$u_{rr} = u_{xx}x_r^2 + u_{yy}y_r^2 + u_{zz}z_r^2 + 2u_{xy}x_r y_r + 2u_{xz}x_r z_r + 2u_{yz}y_r z_r + u_x x_{rr} + u_y y_{rr} + u_z z_{rr}$$
$$u_{\theta\theta} = u_{xx}x_\theta^2 + u_{yy}y_\theta^2 + 2u_{xy}x_\theta y_\theta + u_x x_{\theta\theta} + u_y y_{\theta\theta}$$
$$u_{\phi\phi} = u_{xx}x_\phi^2 + u_{yy}y_\phi^2 + u_{zz}z_\phi^2 + 2u_{xy}x_\phi y_\phi + 2u_{xz}x_\phi z_\phi + 2u_{yz}y_\phi z_\phi + u_x x_{\phi\phi} + u_y y_{\phi\phi} + u_z z_{\phi\phi}$$
$$u_{rr} = u_{xx}(\cos^2\theta\sin^2\phi) + u_{yy}(\sin^2\theta\sin^2\phi) + u_{zz}\cos^2\phi + 2u_{xy}(\cos\theta\sin\theta\sin^2\phi) + 2u_{xz}\cos\theta\sin\phi\cos\phi + 2u_{yz}\sin\theta\sin\phi\cos\phi$$
$$u_{\theta\theta} = u_{xx}r^2\sin^2\theta\sin^2\phi + u_{yy}r^2\cos^2\theta\sin^2\phi - 2u_{xy}r^2\cos\theta\sin\theta\sin^2\phi - u_x r\cos\theta\sin\phi - u_y r\sin\theta\sin\phi$$
$$u_{\phi\phi} = u_{xx}r^2\cos^2\theta\cos^2\phi + u_{yy}r^2\sin^2\theta\cos^2\phi + u_{zz}r^2\sin^2\phi + 2u_{xy}r^2\cos\theta\sin\theta\cos^2\phi - 2u_{xz}r^2\cos\theta\cos\phi\sin\phi - 2u_{yz}r^2\sin\theta\cos\phi\sin\phi - u_x r\cos\theta\sin\phi - u_y r\sin\theta\sin\phi - u_z r\cos\phi$$
$$r^2u_{rr} + u_{\phi\phi} = u_{xx}(r^2\cos^2\theta\,(\sin^2\phi + \cos^2\phi)) + u_{yy}(\mathrm{r}^2\sin^2\theta\,(\sin^2\phi + \cos^2\phi)) + u_{zz}\,\mathrm{r}^2(\cos^2\phi + \sin^2\phi) + 2u_{xy}\left(r^2\cos\theta\sin\theta\,(\sin^2\phi + \cos^2\phi)\right) + 2u_{xz}\,\mathrm{r}^2\cos\theta\sin\phi\cos\phi - 2u_{xz}\,\mathrm{r}^2\cos\theta\sin\phi\cos\phi + 2u_{yz}r^2\sin\theta\sin\phi\cos\phi - 2u_{yz}r^2\sin\theta\cos\phi\sin\phi - u_x r\cos\theta\sin\phi - u_y r\sin\theta\sin\phi - u_z r\cos\phi$$

$$r^2u_{rr} + u_{\phi\phi} = u_{xx}r^2\cos^2\theta + u_{yy}\,\mathrm{r}^2\sin^2\theta + u_{zz}r^2 + 2u_{xy}r^2\cos\theta\sin\theta - u_x r\cos\theta\sin\phi - u_y r\sin\theta\sin\phi - u_z r\cos\phi$$

$$r^2u_{rr} + u_{\phi\phi} + ru_r = u_{xx}r^2\cos^2\theta + u_{yy}\,\mathrm{r}^2\sin^2\theta + u_{zz}r^2 + 2u_{xy}r^2\cos\theta\sin\theta$$

$$u_{\theta\theta} = \sin^2\phi\left(u_{xx}r^2\sin^2\theta + u_{yy}r^2\cos^2\theta - 2u_{xy}r^2\cos\theta\sin\theta - \frac{u_x r\cos\theta}{\sin\phi} - \frac{u_y r\sin\theta}{\sin\phi}\right)$$
$$\frac{1}{\sin^2\phi}u_{\theta\theta} = \left(u_{xx}r^2\sin^2\theta + u_{yy}r^2\cos^2\theta - 2u_{xy}r^2\cos\theta\sin\theta - \frac{u_x r\cos\theta}{\sin\phi} - \frac{u_y r\sin\theta}{\sin\phi}\right)$$

$$r^2u_{rr}+u_{\phi\phi}+ru_r+\frac{1}{\sin^2\phi}u_{\theta\theta}$$
$$=u_{xx}r^2(\cos^2\theta+\sin^2\theta)+u_{yy}\,\mathrm{r}^2(\sin^2\theta+\cos^2\theta)+u_{zz}r^2+2u_{xy}r^2\cos\theta\sin\theta$$
$$-2u_{xy}r^2\cos\theta\sin\theta-\frac{u_xr\cos\theta}{\sin\phi}-\frac{u_yr\sin\theta}{\sin\phi}$$
$$r^2u_{rr}+u_{\phi\phi}+ru_r+\frac{1}{\sin^2\phi}u_{\theta\theta}=r^2(u_{xx}+u_{yy}+u_{zz})-\frac{u_xr\cos\theta}{\sin\phi}-\frac{u_yr\sin\theta}{\sin\phi}$$
$$r^2u_{rr}+u_{\phi\phi}+\frac{1}{\sin^2\phi}u_{\theta\theta}+ru_r=-\frac{u_xr\cos\theta}{\sin\phi}-\frac{u_yr\sin\theta}{\sin\phi}$$
$$u_\phi=u_xr\cos\theta\cos\phi+u_yr\sin\theta\cos\phi-u_zr\sin\phi$$
$$ru_r=u_xr\cos\theta\sin\phi+u_yr\sin\theta\sin\phi+u_zr\cos\phi$$
$$\frac{\cos\phi}{\sin\phi}u_\phi=\frac{u_xr\cos\theta\cos^2\phi}{\sin\phi}+\frac{u_yr\sin\theta\cos^2\phi}{\sin\phi}-u_zr\cos\phi$$
$$\frac{\cos\phi}{\sin\phi}u_\phi+ru_r=\frac{u_xr\cos\theta\cos^2\phi}{\sin\phi}+\frac{u_yr\sin\theta\cos^2\phi}{\sin\phi}+\frac{u_xr\cos\theta\sin^2\phi}{\sin\phi}+\frac{u_yr\sin\theta\sin^2\phi}{\sin\phi}$$
$$\frac{\cos\phi}{\sin\phi}u_\phi+ru_r=\frac{u_xr\cos\theta\,(\cos^2\phi+\sin^2\phi)}{\sin\phi}+\frac{u_yr\sin\theta\,(\cos^2\phi+\sin^2\phi)}{\sin\phi}$$
$$\frac{\cos\phi}{\sin\phi}u_\phi+ru_r=\frac{u_xr\cos\theta}{\sin\phi}+\frac{u_yr\sin\theta}{\sin\phi}$$
$$r^2u_{rr}+u_{\phi\phi}+\frac{1}{\sin^2\phi}u_{\theta\theta}+ru_r=-\left(\frac{\cos\phi}{\sin\phi}u_\phi+ru_r\right)$$
$$r^2u_{rr}+u_{\phi\phi}+\frac{1}{\sin^2\phi}u_{\theta\theta}+2ru_r+\frac{\cos\phi}{\sin\phi}u_\phi=0$$